(a) $\frac{\partial w}{\partial t}+c\frac{\partial w}{\partial x}=e^{2x}$ with $w(x,0)=f(x)$

*(b) $\frac{\partial w}{\partial t}+x\frac{\partial w}{\partial x}=1$ with $w(x,0)=f(x)$

(c) $\frac{\partial w}{\partial t}+t\frac{\partial w}{\partial x}=1$ with $w(x,0)=f(x)$

*(d) $\frac{\partial w}{\partial t}+3t\frac{\partial w}{\partial x}=w$ with $w(x,0)=f(x)$

$$\underbrace{\frac{\partial w}{\partial t}+c\frac{\partial w}{\partial x}}=e^{2x} \qquad w(x_0,0)=f(x_0)$$

already seen this. Idea let $x=x(t)$ to reduce the PDE to an ODE...

only a function of t.

$$\frac{d}{dt}w(x(t),t)=\frac{\partial w}{\partial x}x'(t)+\frac{\partial w}{\partial t}$$

Equation for the characteristics

$$x'(t)=c$$

$$x(t)=ct+x_0$$

Plug in the characteristic

$$\frac{d}{dt}w(x(t),t)=\frac{\partial w}{\partial x}c+\frac{\partial w}{\partial t}=e^{2x}$$

Get this ODE

$$\frac{dw(ct+x_0,t)}{dt}=e^{2(ct+x_0)}$$

Integrate for $t=0$ to t

$$\int_0^t \frac{dw(cs+x_0, s)}{ds}\, ds = \int_0^t e^{2(cs+x_0)}\, ds$$

$$w(ct+x_0, t) - w(x_0, 0) = \frac{1}{2c} e^{2(ct+x_0)} \Big|_0^t$$

$$w(ct+x_0, t) - w(x_0, 0) = \frac{1}{2c} e^{2(ct+x_0)} - \frac{1}{2c} e^{2x_0}$$

$$w(ct+x_0, t) = w(x_0, 0) + \frac{1}{2c} e^{2(ct+x_0)} - \frac{1}{2c} e^{2x_0}$$

$x = ct + x_0$

$x_0 = x - ct$

$$w(x,t) = w(x-ct, 0) + \frac{1}{2c} e^{2x} - \frac{1}{2c} e^{2(x-ct)}$$

$w(x_0, 0) = f(x_0)$

$$w(x,t) = f(x-ct) + \frac{1}{2c} e^{2x} - \frac{1}{2c} e^{2(x-ct)}$$

done...

*(b) $\frac{\partial w}{\partial t} + x\frac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

Let $x = x(t)$

$$\frac{d}{dt} w(x(t), t) = \frac{\partial w}{\partial x} x'(t) + \frac{\partial w}{\partial t}$$

Characteristics:

$$x'(t) = x$$

$$x(t) = x_0 e^t$$

solve

$$x'(t) = x$$

$$\frac{dx}{dt} = x$$

$$\int \frac{dx}{x} = \int dt$$

$$\ln x = t + c$$

$$\boxed{x = e^{t+c} = x_0 e^t}$$

Therefore we obtain the ODE.

$$\frac{d}{dt} w(x_0 e^t, t) = 1$$

$$\int_0^t \frac{d}{ds} w(x_0 e^s, s)\, ds = \int_0^t 1\, ds$$

Thus

$$w(x_0 e^t, t) - w(x_0, 0) = t$$

or

$$w(x_0 e^t, t) = w(x_0, 0) + t$$

$$x = x_0 e^t$$

$$x_0 = x e^{-t}$$

Therefore

$$w(x,t) = w(xe^{-t}, 0) + t$$

The solution is

$$w(x,t) = f(xe^{-t}) + t$$

$w(x,0) = f(x)$

(c) $\dfrac{\partial w}{\partial t} + t\dfrac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

Let $x = x(t)$ then

$$\frac{d}{dt} w(x(t),t) = \frac{\partial w}{\partial x} x'(t) + \frac{\partial w}{\partial t}$$

The characteristics are given by

$$x'(t) = t$$

$$x(t) = \frac{1}{2}t^2 + x_0$$

Along the characteristics

$$\frac{d}{dt} w\left(\frac{1}{2}t^2 + x_0, t\right) = 1$$

Integrate $\int_0^t \cdots dt$ both sides

$$w\Big(\underbrace{\tfrac{1}{2}t^2 + x_0}_{x}, t\Big) - w(x_0, 0) = t$$

$x = \frac{1}{2}t^2 + x_0$

$x_0 = x - \frac{1}{2}t^2$

Therefore

$$w(x,t) = w\left(x - \frac{1}{2}t^2, 0\right) + t$$

Solution

$$\boxed{w(x,t) = f\left(x - \frac{1}{2}t^2\right) + t}$$

*(d) $\frac{\partial w}{\partial t} + 3t\frac{\partial w}{\partial x} = w$ with $w(x,0) = f(x)$

Let $x = x(t)$ then

$$\frac{d}{dt} w(x(t), t) = \frac{\partial w}{\partial x} x'(t) + \frac{\partial w}{\partial t}$$

The characteristics are

$$x'(t) = 3t$$

$$x(t) = \frac{3}{2}t^2 + x_0$$

Along the characteristics the PDE becomes the ODE

$$\frac{d}{dt} w\left(\tfrac{3}{2}t^2 + x_0, t\right) = w\left(\tfrac{3}{2}t^2 + x_0, t\right)$$

$$w\left(\tfrac{3}{2}t^2 + x_0, t\right) = w(x_0, 0)\, e^t$$

$x = \frac{3}{2}t^2 + x_0$

$x_0 = x - \frac{3}{2}t^2$

Thus

$$w(x,t) = w\left(x - \tfrac{3}{2}t^2, 0\right) e^t$$

$w(x,0) = f(x)$

Solution

$$w(x,t) = f\left(x - \tfrac{3}{2}t^2\right) e^t$$

12.6.9. Determine a parametric representation of the solution satisfying $\rho(x,0) = f(x)$:

*(a) $\dfrac{\partial \rho}{\partial t} - \rho^2 \dfrac{\partial \rho}{\partial x} = 3\rho$ (b) $\dfrac{\partial \rho}{\partial t} + \rho \dfrac{\partial \rho}{\partial x} = t$

*(c) $\dfrac{\partial \rho}{\partial t} + t^2 \rho \dfrac{\partial \rho}{\partial x} = -\rho$ (d) $\dfrac{\partial \rho}{\partial t} + \rho \dfrac{\partial \rho}{\partial x} = -x\rho$

cubic in ρ

Consider
$$\frac{\partial \rho}{\partial t} - \rho^2 \frac{\partial \rho}{\partial x} = 3\rho.$$

for nicer version of this calculation skip to last page

Again $x = x(t)$
$$\frac{d}{dt}\rho(x(t),t) = \frac{\partial \rho}{\partial x} x'(t) + \frac{\partial \rho}{\partial t}$$

Thus the characteristics are
$$x'(t) = -\rho(x(t),t)^2$$

How to solve this? Seems stuck...

$$\frac{d\rho}{dt} = 3\rho$$
$$dt = \frac{d\rho}{3\rho}$$
$$\frac{dx}{dt} = -\rho^2$$
$$dt = \frac{dx}{-\rho^2}$$

Try this
$$\frac{d\rho}{dt} = -2\rho,$$
$$\frac{dx}{dt} = -\rho.$$

times written in the equivalent for
$$\frac{d\rho}{-2\rho} = \frac{dx}{-\rho} = dt.$$

$$\frac{d\rho}{3\rho} = \frac{dx}{-\rho^2} = dt$$

$$\frac{d\rho}{3\rho} = \frac{dx}{-\rho^2}$$

means $\rho(x(t),t)$

$$\int_{\rho(0)}^{\rho(t)} -\frac{\rho}{3}\, d\rho = \int_{x(0)}^{x(t)} dx$$

means $f(x_0)$

means $\rho(x(t),t)$

means $\rho(x(t),t)$

$$-\frac{\rho^2}{6}\Bigg|_{\rho(0)}^{\rho(t)} = x\Bigg|_{x(0)}^{x(t)}$$

means $f(x_0)$

means $\rho(x(t),t)$ means $f(x_0)$

$$\frac{-\rho(t)^2 + \rho(0)^2}{6} = x(t) - x_0$$

means $\rho(x(t),t)$ means $f(x_0)$

$$\rho(t)^2 = \rho(0)^2 + 6(x_0 - x(t))$$

means $\rho(x(t),t)$ means $f(x_0)$

$$\frac{d\rho}{dt} = 3\rho \qquad \rho(t) = \rho(0)e^{3t}$$

means $f(x_0)$

$$\rho(0)^2 e^{6t} = \rho(0)^2 + 6(x_0 - x(t))$$

means $f(x_0)$

$$\frac{-\rho(0)^2 e^{6t} + \rho(0)^2}{6} = x(t) - x_0$$

$$x(t) = x_0 + \frac{-\rho(0)^2 e^{6t} + \rho(0)^2}{6}$$

$$\rho(t) = \rho(x(t), t)$$

$$\rho(0) = \rho(x_0, 0) = f(x_0)$$

$$x(t) = x_0 + \frac{-f(x_0)^2 e^{6t} + f(x_0)^2}{6} = x_0 + \frac{f(x_0)^2(1-e^{6t})}{6}$$

$$\rho\Big(\underbrace{x_0 + \frac{f(x_0)^2(1-e^{6t})}{6}}, t\Big) = f(x_0)\, e^{3t}$$

$$x = x_0 + \frac{f(x_0)^2(1-e^{6t})}{6}$$

then solve for x_0 in terms of x

... but difficult... (need to invert f).

Tid

*(a) $\frac{\partial \rho}{\partial t} - \rho^2 \frac{\partial \rho}{\partial x} = 3\rho$

$x = x(t)$

if $x'(t) = -\rho^2$ then

$$\frac{d}{dt}\rho(x(t),t) = \frac{\partial \rho}{\partial x}x'(t) + \frac{\partial \rho}{\partial t} = 3\rho(x(t),t)$$

Thus $\rho(x(t),t) = \rho(x_0, 0)\, e^{3t} = f(x_0)\, e^{3t}$

$$x'(t) = -f(x_0)^2 e^{6t}$$

$$x(t) = -\frac{f(x_0)^2 e^{6t}}{6} + \text{Const}$$

$$x(0) = -\frac{f(x_0)^2}{6} + \text{const} = x_0$$

$$\text{const} = x_0 + \frac{f(x_0)^2}{6}$$

$$x(t) = x_0 + \frac{f(x_0)^2(1-e^{6t})}{6}$$

Answer

$$\rho\left(\underbrace{x_0 + \frac{f(x_0)^2(1-e^{6t})}{6}}_{x}, t\right) = f(x_0)\, e^{3t}.$$

difficult to solve for x_0 in terms of x