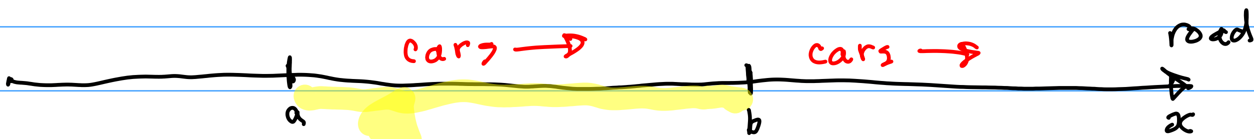
12.6.2 Traffic Flow



$q(x,t)$ = car/hour passing through point x and time t

$\rho(x,t)$ = cars/mile at the point x and time t,

$$\int_a^b \rho(x,t)\,dx = \text{total \# of cars between } a \text{ and } b$$

conservation of cars

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = \underbrace{q(a,t) - q(b,t)}_{\text{looks like the endpoints of an integral}} = \int_b^a \frac{\partial q}{\partial x}\,dx$$

$$\int_a^b \frac{\partial}{\partial t}\rho(x,t)\,dx = -\int_a^b \frac{\partial q}{\partial x}\,dx$$

If this holds for every a and b, then what's inside must be equal...

Therefore we obtain a PDE

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

two variables and one equation...

but they are related

Car velocity. The number of cars per hour passing a place equals the density of cars times the velocity of cars. By introducing $u(x,t)$ as the **car velocity**, we have

$$\boxed{q = \rho u.} \qquad q = \rho u \qquad (12.6.11)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

OH NO! still two variables and one equation...

In the mid-1950s, Lighthill and Whitham and, independently, Richards made a simplifying assumption, namely, that the car velocity depends only on the density, $u = u(\rho)$, with cars slowing down as the traffic density increases (i.e., $du/d\rho \le 0$). For further discussion, the interested reader is referred to Whitham [1999] and Haberman

$u = u(\rho)$ $\quad q = \rho u(\rho)$ $\quad q = q(\rho)$

Chain rule: $\quad \frac{\partial q}{\partial x} = \frac{\partial q}{\partial \rho}\frac{\partial \rho}{\partial x}$

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0$$

$$c(\rho) = \frac{dq}{d\rho}$$

Therefore

initial density of cars...

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \qquad \rho(x_0, 0) = f(x_0)$$

Solve by method of characteristics let $x = x(t)$

$$\frac{d}{dt}\rho(x(t), t) = \frac{\partial \rho}{\partial x}x'(t) + \frac{\partial \rho}{\partial t}$$

If $x'(t) = c(\rho)$ then $\frac{d}{dt}\rho(x(t), t) = 0$

Now $\frac{d}{dt}\rho(x(t),t) = 0$ implies

$$\rho(x(t),t) = \text{const} = \rho(x_0,0) = f(x_0)$$

Plug this back in to solve for the characteristic

$$x'(t) = c(\underbrace{f(x_0)}_{\text{just a constant}})$$

$$x(t) = t\,c(f(x_0)) + x_0$$

Solution implicit form

$$\rho\big(t\,c(f(x_0)) + x_0,\ t\big) = f(x_0)$$

Like to know what $\rho(x,t)$ is ...

$x = t\,c(f(x_0)) + x_0$

try to solve for x_0 in terms of x

to find $\rho(x,t) = f(\,?\,)$

Examples:

Suppose

$c(\rho) = 2\rho$ and $f(x) = \begin{cases} 3 & \text{for } x<0 \\ 4 & \text{for } x>0 \end{cases}$

Then

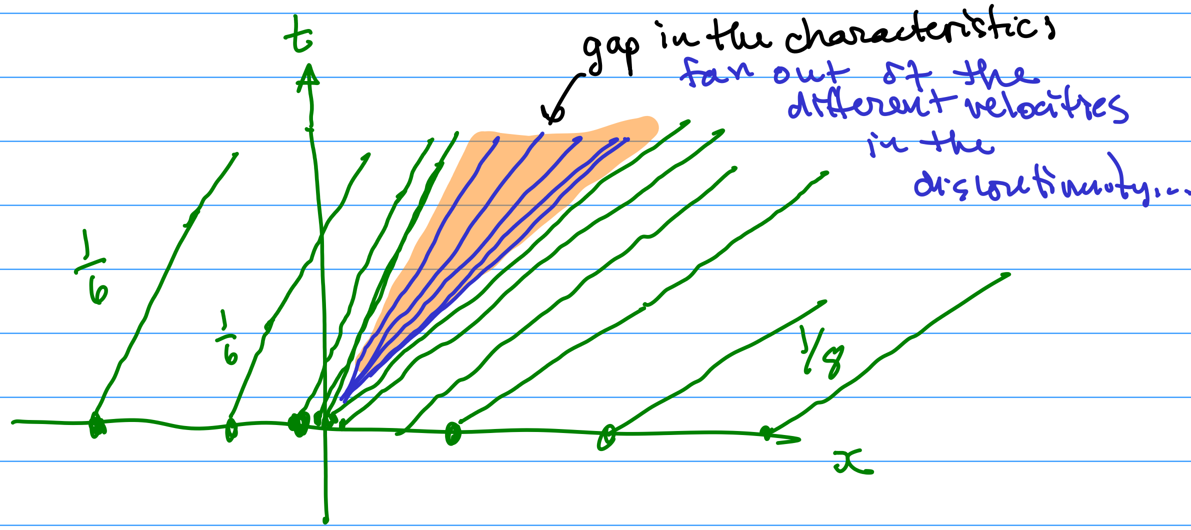
$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$$

and the solution

$$\rho\left(t\,2f(x_0) + x_0,\ t\right) = f(x_0)$$

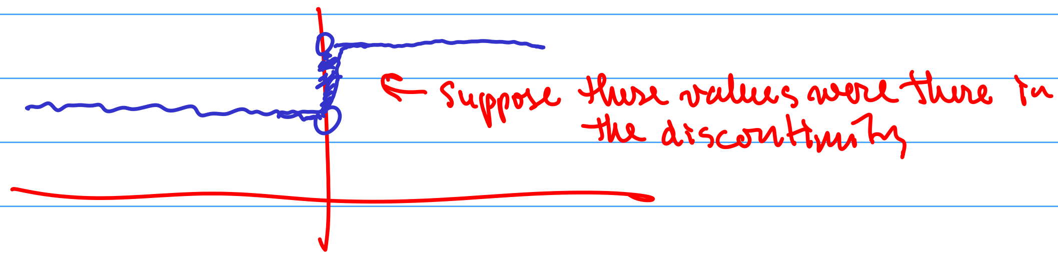
let's think about the characteristics

$$x(t) = 2tf(x_0) + x_0 = \begin{cases} 6t + x_0 & \text{if } x_0 < 0 \\ 8t + x_0 & \text{if } x_0 > 0 \end{cases}$$



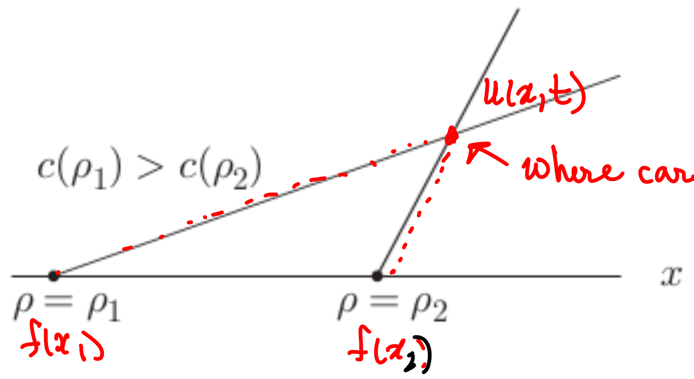
Initial distribution of cars

$$f(x) = \begin{cases} 3 & \text{for } x < 0 \\ 4 & \text{for } x > 0 \end{cases}$$



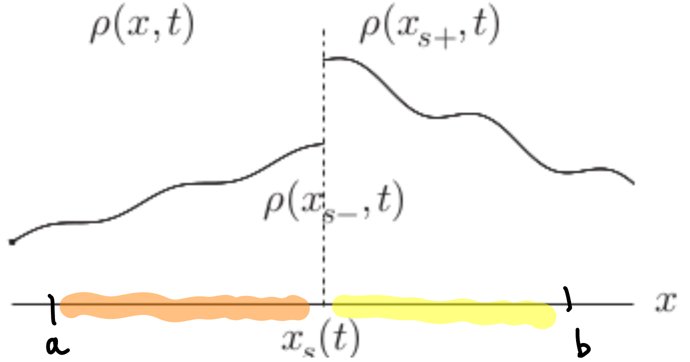
12.6.4 Shock Waves

haracteristics for Quasilinear Partial Differential



$f(x_1) \neq f(x_2)$ but the solution is constant along the characteristics... then the characteristics get close together a jump discontinuity starts to form. At the point where they cross is called a shock.

How does a shock propagate over time?



there is a shock at $x_s(t)$ ...

How does $x_s(t)$ change?

conservation of mass ...

$$\frac{d}{dt}\int_a^b \rho dx = q(a,t) - q(b,t).$$

$$\int_a^b \rho dx = \int_a^{x_s(t)} \rho dx + \int_{x_s(t)}^b \rho dx$$

time dependent

$$\frac{d}{dt}\left(\int_a^{x_s(t)} \rho dx + \int_{x_s(t)}^b \rho dx\right)$$

by chain rule

$$\frac{d}{dt}\int_a^{x_s(t)} \rho dx = \rho(x_s(t),t)\frac{d}{dt}x_s(t) + \int_a^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx$$

$$\frac{d}{dt}G(t) = \frac{d}{dt}\int_a^t g(x)dx = g(t)$$

$$G(t) = \int_a^t g(x)dx$$

$$\frac{d}{dt}G(f(t)) = \frac{d}{dt}\int_a^{f(t)} g(x)dx = G'(f(t))f'(t)$$

$$= g(f(t))f'(t)$$

$$\frac{d}{dt}\int_{x_s(t)}^{b} \rho\, dx = -\frac{d}{dt}\int_{b}^{x_s(t)} \rho\, dx$$

$$= -\left(\rho(x_s(t),t)\, x_s'(t) + \int_{b}^{x_s(t)} \frac{\partial \rho}{\partial t}\, dx\right)$$

$$= -\rho(x_s(t),t)\, x_s'(t) + \int_{x_s(t)}^{b} \frac{\partial \rho}{\partial t}\, dx$$

Therefore

$$\frac{d}{dt}\left(\int_{a}^{x_s(t)} \rho\, dx + \int_{x_s(t)}^{b} \rho\, dx\right)$$

$$= \rho(x_s(t),t)\, \frac{d}{dt} x_s(t) + \int_{a}^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\, dx$$

$$-\rho(x_s(t),t)\, x_s'(t) + \int_{x_s(t)}^{b} \frac{\partial \rho}{\partial t}\, dx$$

However, awa

$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$

then easily

$$\int_{a}^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\, dx = -\int_{a}^{x_s(t)} \frac{\partial q}{\partial x}\, dx = -q(x_s(t)) + q(a)$$

$$\int_{x_s(t)}^{b} \frac{\partial}{\partial t}\rho(x,t)\, dx = -\int_{x_s(t)}^{b} \frac{\partial q}{\partial x}\, dx = -q(b) + q(x_s(t))$$

putting everything together

$$q(a,t) - q(x_{s-},t) + \frac{dx_s}{dt}[\rho(x_{s-},t) - \rho(x_{s+},t)] + q(x_{s+},t) - q(b,t) = q(a,t) - q(b,t).$$

tells $\frac{dx_s}{dt}$ in terms of the other quantities...

Thus...

$$\boxed{\frac{dx_s}{dt} = \frac{q(x_{s-},t) - q(x_{s+},t)}{\rho(x_{s-},t) - \rho(x_{s+},t)} = \frac{[q]}{[\rho]},}$$

tells how the shock moves...