7. [Extra Credit] Use the method of characteristics to solve

$$\frac{\partial \rho}{\partial t} + 5t\frac{\partial \rho}{\partial x} = 3\rho \qquad \text{with} \qquad \rho(x,0) = x^2.$$

Let $x = x(t)$. Then

$$\frac{d\rho(x(t),t)}{dt} = \frac{\partial \rho}{\partial x}x'(t) + \frac{\partial \rho}{\partial t} = 3\rho$$

provided

$$x'(t) = 5t.$$

Solve

$$x(t) = \frac{5}{2}t^2 + x_0$$

also

$$\frac{d\rho(x(t),t)}{dt} = 3\rho(x(t),t)$$

implies

$$\rho(x(t),t) = \rho(x_0,0)e^{3t} = x_0^2 e^{3t}$$

$$\rho(x,0) = x^2.$$

Therefore

$$\rho\left(\frac{5}{2}t^2 + x_0, t\right) = x_0^2 e^{3t}$$

set $x = \frac{5}{2}t^2 + x_0$ and solve for x_0.

$$x_0 = x - \frac{5}{2}t^2$$

Thus

$$\rho(x,t) = \left(x - \frac{5}{2}t^2\right)^2 e^{3t}$$

3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L]$$

subject to the homogeneous boundary conditions

$\phi'(0) = 0$ $\qquad$ $\phi(L) = 0$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

$$u(x,0) = -2\sin\left(\frac{5\pi x}{2L}\right).$$

what if that was a cos?

Separation of variables $u(x,t) = \phi(x)\,h(t)$

$$\phi(x)\,h'(t) = k\,\phi''(x)\,h(t)$$

$$\frac{h'(t)}{k\,h(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

depends only on t

depends only on x

constant since the equation depends neither on x or t.

ODEs are

$$h'(t) = -\lambda k\,h(t) \qquad \text{and} \qquad \phi''(x) = -\lambda\,\phi(x)$$

$$h(t) = c\,e^{-\lambda k t} \qquad\qquad \phi'(0) = 0 \qquad \phi(L) = 0$$

If $\lambda = 0$ then $\varphi''(x) = 0$

$\varphi(x) = ax + b$

$\varphi'(x) = a \quad \varphi'(0) = 0, \; a = 0$

$\varphi(L) = b = 0 \quad$ so $\quad b = 0$

No non-zero eigen function in this case

what if $\lambda = 0$

what if $\lambda < 0$

If $\lambda < 0$ similarly no non-zero eigenfunctions are there. (If time I'll check)

If $\lambda > 0$

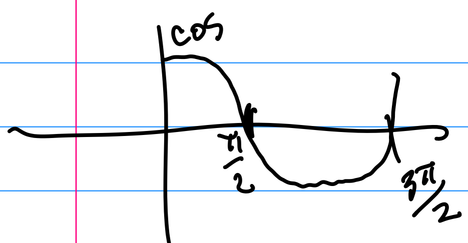
$\varphi(x) = a\cos\sqrt{\lambda}\,x + b\sin\sqrt{\lambda}\,x$

$\varphi'(x) = -a\sqrt{\lambda}\sin\sqrt{\lambda}\,x + b\sqrt{\lambda}\cos\sqrt{\lambda}\,x$

$\varphi'(0) = 0 = b\sqrt{\lambda} = 0 \quad$ so $\quad b = 0 \quad$ since $\lambda \neq 0$.

$\varphi(L) = 0 = a\cos\sqrt{\lambda}\,L \quad$ so $\quad \sqrt{\lambda}\,L = \frac{\pi}{2} + n\pi = (n + \frac{1}{2})\pi$

$\sqrt{\lambda} = \frac{(n+\frac{1}{2})\pi}{L}$



$\lambda = \frac{(n+\frac{1}{2})^2\pi^2}{L^2}$

for $n = 0, 1, \ldots$

$\varphi_n(x) = a\cos\frac{(n+\frac{1}{2})\pi x}{L}$

$$h_n(t) = c\,e^{-\lambda k t} = c\,e^{-\frac{(n+\frac{1}{2})^2\pi^2}{L^2} k t}$$

Use superposition to satisfy the initial cond.

$$u(x,t) = \sum_{n=0}^{\infty} a_n\, e^{-\frac{(n+\frac{1}{2})^2\pi^2}{L^2} k t} \cos\frac{(n+\frac{1}{2})\pi x}{L}$$

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cos\frac{(n+\frac{1}{2})\pi x}{L} = -2\sin\left(\frac{5\pi x}{2L}\right).$$

Therefore by orthogonality

$$\sum_{n=0}^{\infty} a_n \cos\frac{(n+\frac{1}{2})\pi x}{L}\cos\frac{(m+\frac{1}{2})\pi x}{L} = -2\sin\left(\frac{5\pi x}{2L}\right)\cos\frac{(m+\frac{1}{2})\pi x}{L}$$

after integrating over $[0,L]$ we have

$$a_m \frac{L}{2} = \int_0^L -2\sin\left(\frac{5\pi x}{2L}\right)\cos\frac{(m+\frac{1}{2})\pi x}{L}\,dx$$

$$a_m = \frac{2}{L}\int_0^L -2\sin\left(\frac{5\pi x}{2L}\right)\cos\frac{(m+\frac{1}{2})\pi x}{L}\,dx$$

Do I want to simplify this?

If time I'll come back to this, and if can be solved using integration by part or the angle addition formula...

5. Use the method of characteristics to solve

$$\frac{\partial w}{\partial t} + 2\frac{\partial w}{\partial x} = 0 \qquad \text{with} \qquad w(x,0) = \sin x.$$

Let $x = x(t)$

$$\frac{d\, w(x(t),t)}{dt} = \frac{\partial w}{\partial x} x'(t) + \frac{\partial w}{\partial t} = 0$$

provided $x'(t) = 2$

$$x(t) = 2t + x_0$$

Thus

$$w(x(t),t) = \text{const} = w(x_0, 0) = \sin x_0$$

and

$$w(2t + x_0, t) = \sin x_0$$

Let $x = 2t + x_0$ so $x_0 = x - 2t$ and the answer is

$$w(x,t) = \sin(x - 2t).$$

4. For the partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}$$

what ordinary differential equations are implied by the method of separation of variables?

Let $u(x,t) = \phi(x)h(t)$ and substitute

$$\phi(x)h'(t) = k\phi''(x)h(t) - v_0\phi'(x)h(t)$$

Thus

$$\underbrace{\frac{h'(t)}{h(t)}}_{\text{only } t} = \underbrace{\frac{k\phi''(x) - v_0\phi'(x)}{\phi(x)}}_{\text{only } x} = \underbrace{-\lambda}_{\text{so const}}$$

The ODES are

$$h'(t) = -\lambda h(t)$$

and

$$k\phi''(x) - v_0\phi'(x) = -\lambda\phi(x).$$

2. Recall the one-dimensional heat equation with constant thermal properties given by

$$\cancel{c\rho\frac{\partial u}{\partial t}}^{\,0} = K_0\frac{\partial^2 u}{\partial x^2} + Q \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L].$$

Here c is the heat capacity, ρ the density, K_0 the conductivity, Q the rate of production of heat energy and u the temperature. Suppose $L = 2$ and $Q/K_0 = 1$. If the initial and boundary conditions satisfy

$$u(x, 0) = \cos(\pi x) \quad \text{for} \quad x \in [0, 2]$$
$$u(0, t) = 3 \quad \text{and} \quad u(2, t) = 1 \quad \text{for} \quad t > 0,$$

find the equilibrium temperature of the rod obtained as $t \to \infty$.

means u depends only on x not on t

Thus

$$K_0 u''(x) + Q = 0$$

$$u''(x) = -\frac{Q}{K_0} = -1$$

$$u'(x) = -x + a$$

$$u(x) = -\frac{1}{2}x^2 + ax + b$$

general form of the equilibrium temperature

Satisfy boundary

$$u(0, t) = 3 \quad \text{and} \quad u(2, t) = 1$$

$$u(0) = 3 = b \qquad b = 3$$

$$u(2) = 1 = -\frac{1}{2}2^2 + a\cdot 2 + 3$$

$$2a = 1 - 3 + \frac{1}{2}2^2 = 0 \qquad \text{so} \quad a = 0$$

Answer: $\boxed{u(x) = -\frac{1}{2}x^2 + 3}$

6. Consider the wave equation

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u$$

where ρ_0, T_0 and α are constants subject to the homogenous boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

homogeneous in x

Solve the initial value problem if $\alpha < 0$ and

$h(0)=0$

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = f(x).$$

Separation of variables $u(x,t) = \phi(x)h(t)$

$$\rho_0 \phi(x) h''(t) = T_0 \phi''(x) h(t) + \alpha \phi(x) h(t)$$

Thus

$$\frac{\rho_0 h''(t)}{h(t)} = \frac{T_0 \phi''(x) + \alpha \phi(x)}{\phi(x)} = -\lambda$$

The ODEs are

$$h''(t) = \frac{-\lambda}{\rho_0} h(t) \qquad \phi''(x) = -\frac{(\lambda+\alpha)}{T_0}\phi(x)$$

Since $\lambda_n > 0$ then solutions here are also sin and cos.

assume $\lambda+\alpha>0$ to get non-zero eigen functions that satisfy the boundary $\phi(0)=0$ and $\phi(L)=0$

$$h(t) = \alpha \sin\sqrt{\frac{\lambda_n}{\rho_0}}\,t + \beta \cos\sqrt{\frac{\lambda_n}{\rho_0}}\,t$$

$$h(0) = \beta = 0 \qquad \beta = 0$$

$$\phi(x) = a\cos\sqrt{\frac{\lambda+\alpha}{T_0}}\,x + b\sin\sqrt{\frac{\lambda+\alpha}{T_0}}\,x$$

$$\phi(0) = a = 0 \quad \text{so} \quad a = 0$$

$$\phi(L) = b\sin\sqrt{\frac{\lambda+\alpha}{T_0}}\,L = 0$$

$$\sqrt{\frac{\lambda+\alpha}{T_0}}\,L = 2\pi n, \qquad \frac{\lambda+\alpha}{T_0} = \left(\frac{2\pi n}{L}\right)^2$$

$$\text{so} \quad \lambda_n = T_0\left(\frac{2\pi n}{L}\right)^2 - \alpha.$$

$$= T_0\left(\frac{2\pi n}{L}\right)^2 + |\alpha| > 0$$

By superposition

$$u(x,t) = \sum_{n=1}^{\infty} a_n \left(\sin\sqrt{\frac{\lambda_n}{\rho_0}}\, t\right)\left(\sin\sqrt{\frac{\lambda_n + \alpha}{T_0}}\, x\right)$$

$$= \sum_{n=1}^{\infty} a_n \left(\sin\sqrt{\frac{\lambda_n}{\rho_0}}\, t\right)\left(\sin \frac{2\pi n}{L}\, x\right)$$

Solve for the remaining (inhomogeneous) initial cond....

$$\frac{\partial u}{\partial t}(x,0) = f(x).$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a_n \sqrt{\frac{\lambda_n}{\rho_0}} \cos\sqrt{\frac{\lambda_n}{\rho_0}}\, t \left(\sin\frac{2\pi n}{L}\, x\right)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} a_n \sqrt{\frac{\lambda_n}{\rho_0}} \sin\frac{2\pi n x}{L} = f(x)$$

By orthogonality

$$a_n \sqrt{\frac{\lambda_n}{\rho_0}} = \frac{2}{L}\int_0^L f(x) \sin\frac{2\pi n x}{L}\, dx$$

$$a_n = \sqrt{\frac{\rho_0}{\lambda_n}} \cdot \frac{2}{L}\int_0^L f(x) \sin\frac{2\pi n x}{L}\, dx$$

Answer is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \left(\sin\sqrt{\frac{\lambda_n}{\rho_0}}\, t\right)\left(\sin\frac{2\pi n}{L}\, x\right)$$

where

$$a_n = \sqrt{\frac{\rho_0}{\lambda_n}} \cdot \frac{2}{L}\int_0^L f(x) \sin\frac{2\pi n x}{L}\, dx$$

and

$$\lambda_n = T_0\left(\frac{2\pi n}{L}\right)^2 + |\alpha| ,$$

The integral in problem 3:

$$a_m = \frac{2}{L}\int_0^L -2\sin\Big(\underbrace{\frac{5\pi x}{2L}}_{\alpha}\Big)\cos\underbrace{\frac{(m+\frac{1}{2})\pi x}{L}}_{\beta}\,dx$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} = \sin\alpha\cos\beta$$

$$a_m = \frac{-4}{L}\int_0^L \frac{\sin\left(\frac{5\pi x}{2L}+\frac{(m+\frac{1}{2})\pi x}{L}\right)+\sin\left(\frac{5\pi x}{2L}-\frac{(m+\frac{1}{2})\pi x}{L}\right)}{2}\,dx$$

$$= \frac{-4}{L}\int_0^L \frac{\sin\left(\frac{(5+2n+1)\pi x}{2L}\right)+\sin\left(\frac{(5-2n-1)\pi x}{2L}\right)}{2}\,dx$$

$$\int_0^L \sin\frac{(5+2n+1)\pi x}{2L}\,dx = \frac{-2L}{(5+2n+1)\pi}\cos\frac{(5+2n+1)\pi x}{2L}\Big|_0^L$$

Continued after class... dropped the 2 somehow...

$$= \frac{-L}{(3+n)\pi}\big(\cos(3+n)\pi - 1\big)$$

$$= \begin{cases} \dfrac{2L}{(3+n)\pi} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Similarly if $n \neq 2$ then

$$\int_0^L \sin\frac{(5-2n-1)\pi x}{2L}\,dx = \int_0^L \sin\frac{(2-n)\pi x}{L}\,dx$$

$$= \frac{-L}{(2-n)\pi}\cos\frac{(2-n)\pi x}{L}\Big|_0^L$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2L}{(2-n)\pi} & \text{if } n \text{ is odd} \end{cases}$$

Therefore

$$a_n = \frac{-4}{L}\begin{cases} \dfrac{L}{(3+n)\pi} & \text{if } n \text{ is even} \\ \dfrac{L}{(2-n)\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{-4}{(3+n)\pi} & \text{if } n \text{ is even} \\ \dfrac{-4}{(2-n)\pi} & \text{if } n \text{ is odd} \end{cases}$$