HW3 problems 3.2.1bdf, 3.3.1de, 3.3.7, 3.3.17abc, 3.4.5, 3.5.1abc (due Apr 6)

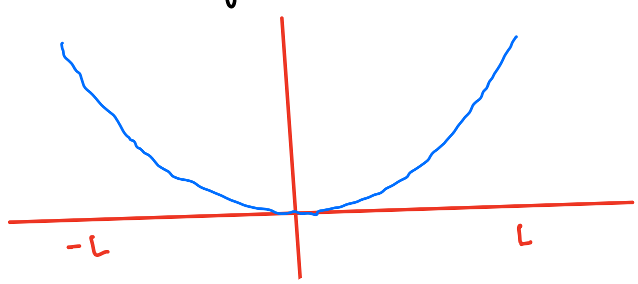
3.2.1. For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$). Compare $f(x)$ to its Fourier series:

(a) $f(x) = 1$
*(b) $f(x) = x^2$
(c) $f(x) = 1 + x$
*(d) $f(x) = e^x$

(e) $f(x) = \begin{cases} x & x < 0 \\ 2x & x > 0 \end{cases}$
*(f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 + x & x > 0 \end{cases}$

(g) $f(x) = \begin{cases} x & x < L/2 \\ 0 & x > L/2 \end{cases}$

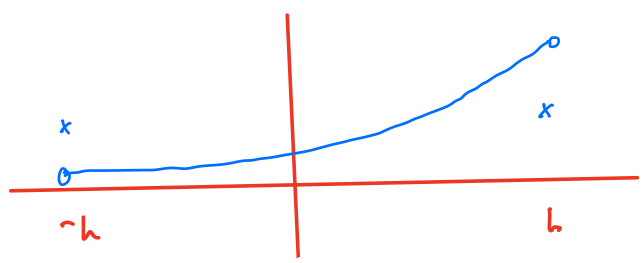
(b) Fourier series converges to



The Fourier series is the same as the original function on [-L, L]

since the periodic extension is piecewise continuous.

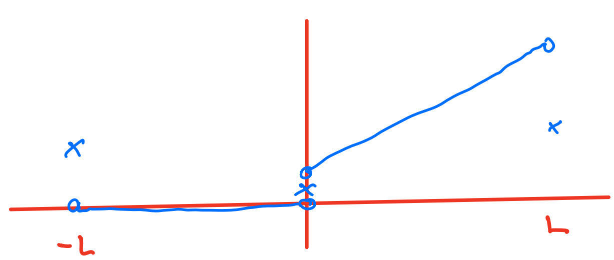
(d) Fourier series converges to



The Fourier series is different than the original function at -L and L

Since the periodic extension is not continuous at -L or L.

(f) Fourier series converges to



The Fourier series is different than the original function at -L, 0 and L

Since the periodic extension is not continuous at -L, 0 and L.

3.3.1. For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$:

(a) $f(x) = 1$

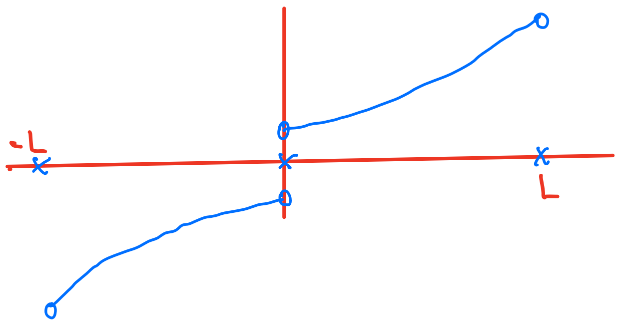
(b) $f(x) = 1 + x$

(c) $f(x) = \begin{cases} x & x < 0 \\ 1 + x & x > 0 \end{cases}$

$*$ **(d)** $f(x) = e^x$

(e) $f(x) = \begin{cases} 2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$

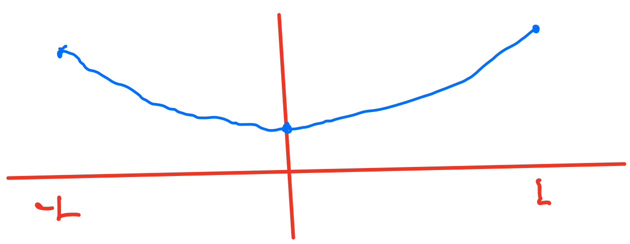
(d) The Fourier sine series is given by the odd extension of the function on $[0, L]$



The sine series differs from the function at 0 and L on the interval $[0, L]$

since the odd extension is not continuous at $-L$, 0 and L.

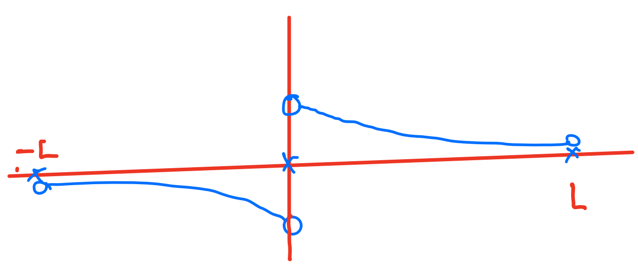
The Fourier cosine series is



The cosine series is equal to the original function on the interval $[0, L]$

since the even extension is piecewise smooth.

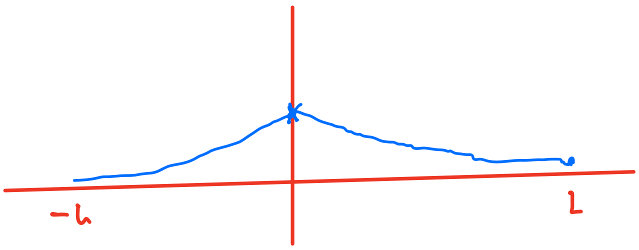
(e) The Fourier sine series is



The sine series differs from the function at 0 and L on the interval $[0, L]$

since the odd extension is discontinuous at $-L$, 0 and L.

The Fourier cosine series is



The cosine series is equal the original function except at 0 where the original function wasn't defined

since the even extension is piecewise smooth everywhere except at 0 where it wasn't defined.

$1 \quad x > L/2$

3.3.7. Show that e^x is the sum of an even and an odd function.

$$e^x = \frac{1}{2}e^x - \frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} + \frac{1}{2}e^x$$

$$= \underbrace{\frac{1}{2}(e^x - e^{-x})}_{\text{odd}} + \underbrace{\frac{1}{2}(e^{-x} + e^x)}_{\text{even}} = f_o(x) + f_e(x)$$

where f_o is the odd function

$$f_o(x) = \frac{1}{2}(e^x - e^{-x})$$

and f_e is the even function

$$f_e(x) = \frac{1}{2}(e^x + e^{-x})$$

3.3.17. Consider

$$\int_0^1 \frac{dx}{1+x^2}.$$

(a) Evaluate explicitly.

(b) Use the Taylor series of $1/(1+x^2)$ (itself a geometric series) to obtain an infinite series for the integral.

(c) Equate part (a) to part (b) in order to derive a formula for π.

(a) Evaluate

$$\int_0^1 \frac{dx}{1+x^2} = \arctan x \Big|_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}$$

(b) Since

$$\begin{array}{r|l} & 1 - x^2 + x^4 \cdots \\ \hline 1+x^2 & 1 \\ & \underline{1+x^2} \\ & -x^2 \\ & \underline{-x^2 - x^4} \\ & x^4 \\ & \underline{x^4 + x^6} \\ & -x^6 \\ & \vdots \end{array}$$

we obtain that $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$

It follows that

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^1 (1 - x^2 + x^4 - x^6 + x^8 - \cdots)\,dx$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \cdots \Big|_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

(c) a formula for π is

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$$

entiated term by term.

3.4.5. Using (3.3.13) determine the Fourier cosine series of $\sin \pi x/L$.

3.4.6. There are some things wrong in the following demonstration. Find the mistakes and

Recall

$$B_n = \frac{2}{L}\int_0^L \cos\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)} & n \text{ even.} \end{cases} \qquad (3.3.13)$$

The cosine series of f is

$$\sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \text{ for } n>0.$$

When $f(x) = \sin\frac{\pi x}{L}$ we obtain

$$a_0 = \frac{1}{L}\int_0^L \sin\frac{\pi x}{L}\,dx = -\frac{1}{L}\frac{L}{\pi}\cos\frac{\pi x}{L}\Big|_0^L = \frac{2}{\pi}$$

and for $n>0$ that

$$a_n = \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

It's unclear to me how to directly use (3.3.13) as the sin and cos are interchanged. I procede by analogy and use the angle addition formula to simplify as

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \hline \sin(\alpha+\beta)+\sin(\alpha-\beta) &= 2\sin\alpha\cos\beta \end{aligned}$$

Therefore

$$a_n = \frac{2}{L}\int_0^L \frac{\sin\left(\frac{\pi x}{L}+\frac{n\pi x}{2}\right)+\sin\left(\frac{\pi x}{L}-\frac{n\pi x}{L}\right)}{2}\,dx$$

If $n = 2k+1$ is odd, then

$$a_{2k+1} = \frac{1}{L}\int_0^L \left(\sin\left(\frac{(2k+2)\pi x}{L}\right) + \sin\left(\frac{-2k\pi x}{L}\right)\right) dx$$

If $k=0$ then

$$a_1 = \frac{1}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right) dx = \frac{-1}{2\pi}\cos\frac{2\pi x}{L}\Big|_0^L = 0$$

and in $k>0$ then

$$a_{2k+1} = \frac{-1}{(2k+2)\pi}\cos\frac{(2k+2)\pi x}{L}\Big|_0^L + \frac{-1}{2k\pi}\cos\frac{2k\pi x}{L}\Big|_0^L = 0$$

If $n = 2k > 0$ is even, then

$$a_{2k} = \frac{1}{L}\int_0^L \left(\sin\left(\frac{(2k+1)\pi x}{L}\right) + \sin\left(\frac{(-2k+1)\pi x}{L}\right)\right) dx$$

$$= \frac{-1}{(2k+1)\pi}\cos\frac{(2k+1)\pi x}{L}\Big|_0^L + \frac{-1}{(-2k+1)\pi}\cos\frac{(-2k+1)\pi x}{L}\Big|_0^L$$

$$= \frac{2}{\pi}\left(\frac{1}{1+2k} + \frac{1}{1-2k}\right) = \frac{2}{\pi}\left(\frac{1-2k+1+2k}{1-4k^2}\right)$$

$$= \frac{4}{\pi(1-4k^2)} = \frac{-4}{\pi(n^2-1)}.$$

Therefore

$$a_n = \begin{cases} \dfrac{-4}{\pi(n^2-1)} & \text{if } n>0 \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

and it follows that

$$\sin\frac{\pi x}{L} \sim \frac{2}{\pi} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{-4}{\pi(n^2-1)}\cos\frac{n\pi x}{L}$$

$$= \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{-4}{\pi(4k^2-1)}\cos\frac{2k\pi x}{L}.$$

3.5.1. Consider

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \qquad (3.5.12)$$

(a) Determine b_n from (3.3.11), (3.3.12), and (3.5.6).

(b) For what values of x is (3.5.12) an equality?

***(c)** Derive the Fourier cosine series for x^3 from (3.5.12). For what values of x will this be an equality?

(a) Recall

$$x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \qquad -L < x < L. \qquad (3.3.11)$$

where

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx = \frac{2}{L}\int_0^L x \sin \frac{n\pi x}{L}\, dx = \frac{2L}{n\pi}(-1)^{n+1}. \qquad (3.3.12)$$

Also

$$\frac{x^2}{2} = \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin \frac{\pi x}{L} + \frac{\sin 3\pi x/L}{3^3} + \frac{\sin 5\pi x/L}{5^3} + \cdots\right). \qquad (3.5.6)$$

Substituting (3.3.1) into (3.56) yields

$$\frac{x^2}{2} = \frac{L}{2}\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} - \frac{4L^2}{\pi^3}\left(\sin \frac{\pi x}{L} + \frac{1}{3^3}\sin \frac{3\pi x}{L} + \frac{1}{5^3}\sin \frac{5\pi x}{L} + \cdots\right)$$

$$= \frac{L}{2}\sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{L} - \frac{4L^2}{\pi^3}\sum_{n \text{ odd}} \frac{1}{n^3}\sin \frac{n\pi x}{L}$$

$$= \frac{L^2}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin \frac{n\pi x}{L} - \frac{L^2}{\pi}\sum_{n \text{ even}} \frac{1}{n}\sin \frac{n\pi x}{L} - \frac{4L^2}{\pi^3}\sum_{n \text{ odd}} \frac{1}{n^3}\sin \frac{n\pi x}{L}$$

$$= \frac{L^2}{\pi}\sum_{n \text{ odd}} \left(\frac{1}{n} - \frac{4}{\pi^2 n^3}\right)\sin \frac{n\pi x}{L} - \frac{L^2}{\pi}\sum_{n \text{ even}} \frac{1}{n}\sin \frac{n\pi x}{L}$$

Therefore

$$b_n = \begin{cases} 2\frac{L^2}{\pi}\left(\frac{1}{n} - \frac{4}{\pi^2 n^3}\right) & \text{for } n \text{ odd} \\ -2\frac{L^2}{\pi}\frac{1}{n} & \text{for } n \text{ even} \end{cases}$$

(b) Since the odd extension of x^2 is smooth everywhere except $x=L$ then (3.5.12) is an equality on $[0,L)$, at $x=L$ it will converge to the average $\frac{0+L^2}{2} = \frac{L^2}{2}$.

(c) The Fourier cosine series may be obtained by integrating term by term. Thus,

$$\frac{x^3}{9} \sim \sum_{n=1}^{\infty} \int_0^x b_n \sin\frac{n\pi s}{L}\, ds = \sum_{n=1}^{\infty} \frac{-b_n L}{n\pi} \cos\frac{n\pi s}{L}\Big|_0^x$$

$$= \sum_{n=1}^{\infty} \frac{-b_n L}{n\pi}\left(\cos\frac{n\pi x}{L} - 1\right) = \sum_{n=1}^{\infty} \frac{b_n L}{n\pi} + \sum_{n=1}^{\infty} \frac{-b_n L}{n\pi} \cos\frac{n\pi x}{L}$$

Therefore

$$x^3 = \sum_{n=0}^{\infty} a_0 \cos\frac{n\pi x}{L}$$

where

$$a_0 = 3\sum_{n=1}^{\infty} \frac{b_n L}{n\pi} \quad \text{and} \quad a_n = \frac{-3 b_n L}{n\pi} \quad \text{for } n>0$$

Here again

$$b_n = \begin{cases} 2\frac{L^2}{\pi}\left(\frac{1}{n} - \frac{4}{\pi^2 n^3}\right) & \text{for } n \text{ odd} \\ -2\frac{L^2}{\pi}\frac{1}{n} & \text{for } n \text{ even} \end{cases}.$$

simplifying, obtains

$$a_0 = 6\frac{L^3}{\pi^2}\left(\sum_{n\ \mathrm{odd}} \left(\frac{1}{n^2} - \frac{4}{\pi^2 n^4}\right) - \sum_{n\ \mathrm{even}} \frac{1}{n^2}\right)$$

$$= 6\frac{L^3}{\pi^2}\left(\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^{n+1} - \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}\right)$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^{n+1} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots = \frac{\pi^2}{12}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$$

as seen by Maple (Wolfram Alpha also given same)

```
> sum(1/n^2*(-1)^(n+1),n=1..infinity);
                                              2
                                            Pi
                                            ---
                                            12

> sum(1/(2*k+1)^4,k=0..infinity);
                                              4
                                            Pi
                                            ---
                                            96
```

see below for an easier way to find a_0

Therefore

$$a_0 = \frac{6L^3}{\pi^2}\left(\frac{\pi^2}{12} - \frac{4}{\pi^2}\frac{\pi^4}{96}\right) = 6L^3\left(\frac{1}{12} - \frac{1}{24}\right) = \frac{L^3}{4}$$

Note, a simpler way to find a_0 is from the definition

$$a_0 = \frac{1}{L}\int_0^L x^3\,dx = \frac{1}{4L}\,x^4\Big|_0^L = \frac{L^3}{4}.$$

It follows that

$$x^3 = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where

$$a_0 = \frac{L^3}{4}$$

and for $n>0$ that

$$a_n = \begin{cases} 6\frac{L^3}{\pi^2}\left(\frac{1}{n^2} - \frac{4}{\pi^2 n^4}\right) & \text{for } n \text{ odd} \\ -6\frac{L^3}{\pi^2}\frac{1}{n^2} & \text{for } n \text{ even.} \end{cases}$$