Fourier Series:

The Fourier series of a continuous function $u(x,t)$ (depending on a parameter t)

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty}\left[a_n(t)\cos\frac{n\pi x}{L} + b_n(t)\sin\frac{n\pi x}{L}\right]$$

can be differentiated term by term with respect to the parameter t, yielding

$$u_t(x,t) = \frac{\partial}{\partial t}u(x,t) \sim a_0'(t) + \sum_{n=1}^{\infty}\left[a_n'(t)\cos\frac{n\pi x}{L} + b_n'(t)\sin\frac{n\pi x}{L}\right]$$

if $\partial u/\partial t$ is $\boxed{\text{piecewise}}$ smooth.

for simplicity $u_t(x,t)$ is smooth

why is this true?

periodic function in x with period $2L$

$$a_0(t) = \frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx \qquad a_n(t) = \frac{1}{L}\int_{-L}^{L} u(x,t)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n(t) = \frac{1}{L}\int_{-L}^{L} u(x,t)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier series for $u_t(x,t)$

$$u_t(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty}\left(\alpha_n(t)\cos\frac{n\pi x}{L} + \beta_n(t)\sin\frac{n\pi x}{L}\right)$$

$$\alpha_0(t) = \frac{1}{2L}\int_{-L}^{L} u_t(x,t)\,dx \qquad \alpha_n(t) = \frac{1}{L}\int_{-L}^{L} u_t(x,t)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\beta_n(t) = \frac{1}{L}\int_{-L}^{L} u_t(x,t)\sin\left(\frac{n\pi x}{L}\right)dx$$

Need to show that

$\alpha_0(t) = a_0'(t)$ $\quad$ $\alpha_n(t) = a_n'(t)$ and $\beta_n(t) = b_n'(t)$

first

combine

$$a_0'(t) = \frac{d}{dt}\frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx = \lim_{h\to 0}\frac{\frac{1}{2L}\int_{-L}^{L} u(x,t+h)\,dx - \frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx}{h}$$

goal is to interchange $\frac{d}{dt}$ with $\int_{-L}^{L}$.

when can you do this?

$$\ldots = \lim_{h\to 0} \frac{1}{2L}\int_{-L}^{L} \frac{u(x,t+h)-u(x,t)}{h}\,dx$$

to estimate difference quotients use the mean value theorem

Mean value theorem:

$$\frac{u(x,t+h)-u(x,t)}{h} = u_t(x,t+\xi)$$

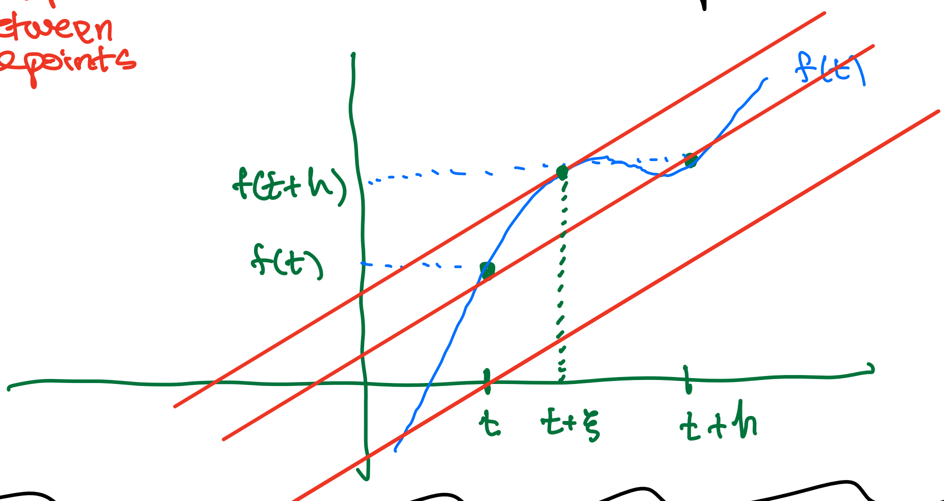
for some ξ between 0 and h
note ξ depends on t and x.

Review

$$\frac{f(t+h)-f(t)}{h} = f'(t+\xi)$$

for some ξ between 0 and h. Note ξ depends on t.

Slope between the points



$$a_0'(t) = \lim_{h\to 0} \frac{1}{2L}\int_{-L}^{L} \frac{u(x,t+h)-u(x,t)}{h}\,dx$$

$$= \lim_{h\to 0} \frac{1}{2L}\int_{-L}^{L} u_t(x,t+\xi(t,x))\,dx$$

for some ξ between 0 and h
note ξ depends on t and x.

Consider $u_t(x,t)$ note, this function is smooth by hypothesis... and on the closed set

$$(x,t) \in [-L,L] \times [t-|h|, t+|h|]$$

this function has a maximum.

Thus $|u_t(x,t)| \leq \text{Bound} = B$ for

$$(x,t) \in [-L,L] \times [t-|h|, t+|h|]$$

$$Q_0'(t) = \lim_{h \to 0} \frac{1}{2L} \int_{-L}^{L} u_t(x, t+\xi(t,x))\, dx$$

$$= \lim_{h \to 0} \frac{1}{2L} \int_{-L}^{L} \big(u_t(x, t+\xi(t,x)) - u_t(x,t) + u_t(x,t) \big)\, dx$$

Now, by the mean value theorem again

$$\int_{-L}^{L} \big(u_t(x, t+\xi(t,x)) - u_t(x,t) \big)\, dx$$

$$= \int_{-L}^{L} u_{tt}(x, t+\psi(t,x))\, \psi(t,x)\, dx$$

where $\psi(t,x)$ is between 0 and $\xi(t,x)$ and so between 0 and h

Consider $u_{tt}(x,t)$ note, this function is continuous by hypothesis... and on the closed set

$$(x,t) \in [-L,L] \times [t-|h|, t+|h|]$$

this function has a maximum.

Thus $|u_{tt}(x,t)| \leq \text{Bound} = C$ for

$$(x,t) \in [-L,L] \times [t-|h|, t+|h|]$$

$$\left|\int_{-L}^{L}\left(u_t(x,t+\xi(t,x))-u_t(x,t)\right)dx\right|$$

$$\leq \int_{-L}^{L}\left|u_{tt}(x,t+\varphi(t,x))\varphi(t,x)\right|dx$$

$$\leq \int_{-L}^{L} c\,|\varphi(t,x)|\,dx \leq \int_{-L}^{L} C|h|\,dx = 2LC|h|$$

$$\to 0 \quad \text{as } h\to 0$$

Therefore,

$$a_0'(t) = \lim_{h\to 0}\frac{1}{2L}\int_{-L}^{L}\left(u_t(x,t+\xi(t,x))-u_t(x,t)+u_t(x,t)\right)dx$$

or

$$a_0'(t)-\lim_{h\to 0}\frac{1}{2L}\int_{-L}^{L}u_t(x,t)\,dx = \lim_{h\to 0}\frac{1}{2L}\int_{-L}^{L}\left(u_t(x,t+\xi(t,x))-u_t(x,t)\right)dx$$

$$=0$$

Therefore

$$a_0'(t) = \lim_{h\to 0}\frac{1}{2L}\int_{-L}^{L}u_t(x,t)\,dx = \alpha_0(t)$$

note for $a_n'(t)=\alpha_n(t)$ and $b_n'(t)=\beta_n(t)$ the only difference is the $\cos\frac{n\pi x}{L}$ or $\sin\frac{n\pi x}{L}$ respectively. since these are nice smooth function as well multiplying them in doesn't hurt and the same argument works...

Summary of argument:

- a continuous function on a product of closed intervals has a maximum so it's bounded...
- Mean value theorem (twice).