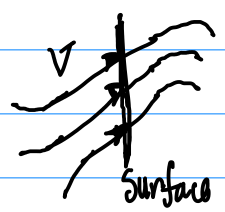
1.5.2. For conduction of thermal energy, the heat flux vector is $\boldsymbol{\phi} = -K_0 \nabla u$. If in addition the molecules move at an average velocity $\boldsymbol{V}$, a process called **convection**, then briefly explain why $\boldsymbol{\phi} = -K_0 \nabla u + c\rho u \boldsymbol{V}$. Derive the corresponding equation for heat flow, including both conduction and convection of thermal energy (assuming constant thermal properties with no sources).

The energy flux ϕ is the energy passing through a surface per unit time. Thus the units of ϕ are

$$[\phi] = \frac{[E]}{[L^2][T]}$$

where $[E]$ is dimensions of energy, $[L]$ is dimensions of length and $[T]$ stands for dimensions of time

If the molecules are moving at velocity V then the energy in those molecules will pass through any surface intersected by the direction of the velocity.



Energy passing through a surface

The energy density is $c\rho u$ where c is the heat capacity of the material and ρ its density. To obtain a flux it is enough to multiply this by the velocity, since dimensionally

$$[c\rho u] = \frac{[E]}{[L^3]} \quad \text{and} \quad [V] = \frac{[L]}{[T]}$$

implies

$$[c\rho u V] = \frac{[E]}{[L^3]}\frac{[L]}{[T]} = \frac{[E]}{[L^2][T]}$$

which is a flux.

To obtain the resulting equation for heat flow plug this flux φ into the balance of energy equation

$$c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \varphi + Q$$

By vector calculus identites

$$\nabla \cdot (-K_0 \nabla u + c\rho u V) = -K_0 \nabla^2 u + c\rho \nabla \cdot (uV)$$

$$= -K_0 \nabla^2 u + c\rho \nabla u \cdot V + c\rho u \nabla \cdot V$$

Consequently

$$c\rho \frac{\partial u}{\partial t} + c\rho \nabla u \cdot V + c\rho u \nabla \cdot V = K_0 \nabla^2 u + Q$$

No sources means $Q=0$. Consequently setting the diffusivity

$$k = \frac{K_0}{c\rho}$$

yields

$$\frac{\partial u}{\partial t} + \nabla u \cdot V + u \nabla \cdot V = k \nabla^2 u.$$

2.3.1. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

* **(a)** $\dfrac{\partial u}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)$

Let $u(r,t) = g(t)\phi(r)$. Upon substituting

$$g'(t)\phi(r) = g(t)\frac{k}{r}\frac{\partial}{\partial r}\left(r\phi'(r)\right)$$

Consequently

$$\frac{g'(t)}{kg(t)} = \frac{1}{\phi(r)\, r}\frac{d}{dr}\left(r\phi'(r)\right)$$

Since the left side is independent of r and the right is independent of t, both must be constant. Denote that constant by $-\lambda$ to obtain

$$g'(t) = -\lambda k g(t) \quad \text{and} \quad \frac{1}{r}\frac{d}{dr}\left(r\phi'(r)\right) = -\lambda\phi(r)$$

(d) $\dfrac{\partial u}{\partial t} = \dfrac{k}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)$

Again let $u(t,r) = g(t)\phi(r)$ so that

$$g'(t)\phi(r) = g(t)\frac{k}{r^2}\frac{d}{dr}\left(r^2\phi'(r)\right)$$

and consequently

$$\frac{g'(t)}{kg(t)} = \frac{1}{\phi(r)r^2}\frac{d}{dr}\left(r^2\phi'(r)\right)$$

Since the left side is independent of r and the right is independent of t, both must be constant. Denote that constant by $-\lambda$ to obtain

$$g'(t) = -\lambda k\, g(t) \qquad \text{and} \qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2 \varphi'(r)\right) = -\lambda \varphi(r)$$

2.3.2. Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0, \lambda = 0, \lambda < 0$). You may assume that the eigenvalues are real.

***(b)** $\phi(0) = 0$ and $\phi(1) = 0$

Case $\lambda > 0$. Then the general solution is

$$\phi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

Solving for the boundary conditions as

$$\phi(0) = A\cos(0) + B\sin(0) = A = 0$$

$$\phi(1) = B\sin(\sqrt{\lambda}) = 0$$

and noting $B \neq 0$ for ϕ to be a non-zero function, we obtain that

$\sin(\sqrt{\lambda}) = 0$ or that $\sqrt{\lambda} = n\pi$ where $n = 1, 2, \ldots$

In this case we obtain the eigenfunctions

$$\phi(x) = B\sin[n\pi x] \quad \text{with } \lambda = (n\pi)^2.$$

Case $\lambda = 0$. Then the general solution is

$$\phi(x) = Ax + B$$

Solving for the boundary conditions

$$\phi(0) = A0 + B = B = 0$$

and $\phi(1) = A = 0$

in which case $\phi = 0$ and so there is no eigenfunction that corresponds to $\lambda = 0$

Case $\lambda < 0$. Then the general solution is

$$\phi(x) = Ae^{\sqrt{|\lambda|}\,x} + Be^{-\sqrt{|\lambda|}\,x}$$

Again solving for the boundary conditions obtains

$$\phi(0) = Ae^{0} + Be^{-0} = A + B = 0 \qquad \text{so} \qquad B = -A$$

$$\phi(1) = Ae^{\sqrt{|\lambda|}} - Ae^{-\sqrt{|\lambda|}} = A\left(e^{\sqrt{|\lambda|}} - e^{-\sqrt{|\lambda|}}\right) = 0$$

Therefore

$$A = \frac{0}{e^{\sqrt{|\lambda|}} - e^{-\sqrt{|\lambda|}}} = 0 \qquad \text{and} \qquad B = 0$$

and there are no eigenfunctions that correspond to $\lambda < 0$

***(d)** $\phi(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$

Case $\lambda > 0$. Then the general solution is

$$\phi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

and consequently

$$\phi'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$$

Solving for the boundary conditions as

$$\phi(0) = A\cos(0) + B\sin(0) = A = 0$$

$$\phi'(L) = B\sqrt{\lambda}\cos(\sqrt{\lambda}\,L) = 0$$

implies since $B \neq 0$ that

$\cos(\sqrt{\lambda}L)=0$ or that $\sqrt{\lambda}L=\frac{\pi}{2}+n\pi$ for $n=0,1,\ldots$

In this case we obtain the eigenfunctions

$$\varphi(x)=B\sin\left(\frac{(n+\frac{1}{2})\pi x}{L}\right) \text{ with } \lambda=\left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2 \text{ for } n=0,1,\ldots$$

Case $\lambda=0$. Then the general solution is

$$\varphi(x)=Ax+B$$

and consequently

$$\varphi'(x)=A$$

Solving for the boundary conditions

$$\varphi(0)=A0+B=B=0$$

and $\varphi'(L)=A=0$

in which case $\varphi=0$ and so there is no eigenfunction that corresponds to $\lambda=0$

Case $\lambda<0$. Then the general solution is

$$\varphi(x)=Ae^{\sqrt{|\lambda|}x}+Be^{-\sqrt{|\lambda|}x}$$

and consequently

$$\varphi'(x)=A\sqrt{|\lambda|}e^{\sqrt{|\lambda|}x}-B\sqrt{|\lambda|}e^{-\sqrt{|\lambda|}x}.$$

again solving for the boundary conditions obtains

$$\varphi(0) = Ae^{0} + Be^{-0} = A+B = 0 \quad \text{so} \quad B=-A$$

$$\varphi'(L) = A\sqrt{|\lambda|}e^{\sqrt{|\lambda|}L} + A\sqrt{|\lambda|}e^{-\sqrt{|\lambda|}} = A(e^{\sqrt{|\lambda|}} + e^{-\sqrt{|\lambda|}}) = 0$$

Therefore $$A = \frac{0}{e^{\sqrt{|\lambda|}} + e^{-\sqrt{|\lambda|}}} = 0 \quad \text{and} \quad B=0$$

and there are no eigenfunctions that correspond to $\lambda < 0$

2.3.3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

(a) $u(x,0) = 6\sin\frac{9\pi x}{L}$

Separation of variables as $u(x,t) = \phi(x)g(t)$ yields

$$\phi(x)g'(t) = k\,\phi''(x)g(t)$$

so that

$$\frac{g'(t)}{k\,g(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

where λ is a constant. It follows that

$$g'(t) = -\lambda k\,g(t) \qquad \text{and} \qquad \phi''(x) = -\lambda\phi(x)$$

The general solutions to the above ordinary differential equations is

$$g(t) = C e^{-\lambda k t} \qquad \text{and} \qquad \phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

The boundary conditions for ϕ are homogenous. In particular

$$\phi(0) = A\cos 0 + B\sin 0 = A = 0$$

$$\phi(L) = B\sin\sqrt{\lambda}L = 0$$

Since $B \neq 0$ then $\sin\sqrt{\lambda}L = 0$ implies $\sqrt{\lambda}L = n\pi$ for $n = 1, 2, \ldots$

Consequently $\lambda = \left(\frac{n\pi}{L}\right)^2$

The superposition principle then leads to a general solution to the heat equation of the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right)$$

which satisfies the boundary conditions.

To satisfy the initial condition we need

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 6\sin\left(\frac{9\pi x}{L}\right).$$

From the orthogonality of the sine functions we imediatly infer that the only non-zero coefficient B_n occurs when $n=9$ and that $B_9 = 6$.

Therefore

$$u(x,t) = 6 e^{-\left(\frac{9\pi}{L}\right)^2 kt} \sin\left(\frac{9\pi x}{L}\right)$$

$*$ **(c)** $u(x,0) = 2\cos\frac{3\pi x}{L}$

The general solution is again

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right)$$

but this time we need

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 2\cos\frac{3\pi x}{L}$$

By orthogonality

$$B_n = \frac{2}{L}\int_0^L 2\cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

Recall

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

so that

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$

Substituting yields

$$B_n = \frac{2}{L}\int_0^L \left(\sin\left(\frac{(3+n)\pi x}{L}\right) - \sin\left(\frac{(3-n)\pi x}{L}\right)\right)dx$$

$$= \frac{2}{L}\,\frac{-L}{(3+n)\pi}\cos\left(\frac{(3+n)\pi x}{L}\right)\Bigg|_0^L + \begin{cases} 0 & \text{if } n=3 \\ \frac{2}{L}\,\frac{L}{(3-n)\pi}\cos\left(\frac{(3-n)\pi x}{L}\right)\Bigg|_0^L & \text{otherwise} \end{cases}$$

$$= \frac{-2}{(3+n)\pi}\left(\cos(3+n)\pi - 1\right) + \begin{cases} 0 & \text{if } n=3 \\ \frac{2}{(3-n)\pi}\left(\cos(3-n)\pi - 1\right) & \text{otherwise} \end{cases}$$

$$= \frac{-2}{(3+n)\pi}\left((-1)^{(3+n)} - 1\right) + \begin{cases} 0 & \text{if } n=3 \\ \frac{2}{(3-n)\pi}\left((-1)^{(3-n)} - 1\right) & \text{otherwise} \end{cases}$$

Now, since $(-1)^{(3+n)} - 1 = \begin{cases} 0 & \text{when } n \text{ is odd} \\ -2 & \text{when } n \text{ is even} \end{cases}$

and $(-1)^{(3-n)} - 1 = \begin{cases} 0 & \text{when } n \text{ is odd} \\ -2 & \text{when } n \text{ is even} \end{cases}$

it follows that

$$B_n = \begin{cases} \frac{4}{(3+n)\pi} - \frac{4}{(3-n)\pi} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$$= \begin{cases} \frac{4}{\pi}\left(\frac{-2n}{9-n^2}\right) & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$$= \begin{cases} \frac{8n}{\pi(n^2-9)} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

Consequently

$$u(x,t) = \sum_{n \text{ even}} \frac{8n}{\pi(n^2-9)} e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{m=1}^{\infty} \frac{16m}{\pi(4m^2-9)} e^{-\left(\frac{2m\pi}{L}\right)^2 kt} \sin\left(\frac{2m\pi x}{L}\right)$$

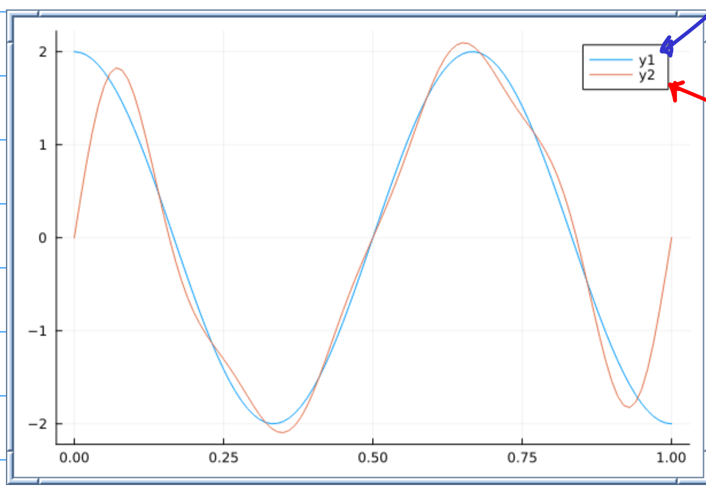
I used Julia https://julialang.org/ to numerically check the above expression using the commands

```
using Plots

b(m)=16*m/(pi*(4*m^2-9))
g(m,t)=exp(-(2*m*pi/L)^2*k*t)
phi(m,x)=sin(2*m*pi*x/L)
u(x,t)=sum([b(m)*g(m,t)*phi(m,x) for m=1:5])
u0(x)=u(x,0)

L=1
k=1
xs=0:0.01:1
plot(xs,2*cos.(3*pi*xs/L))
plot!(xs,u0.(xs))
```

to sum the first 5 terms of the series approximation to the initial condition and compare that to $2\cos\left(\frac{3\pi x}{L}\right)$. The graph is

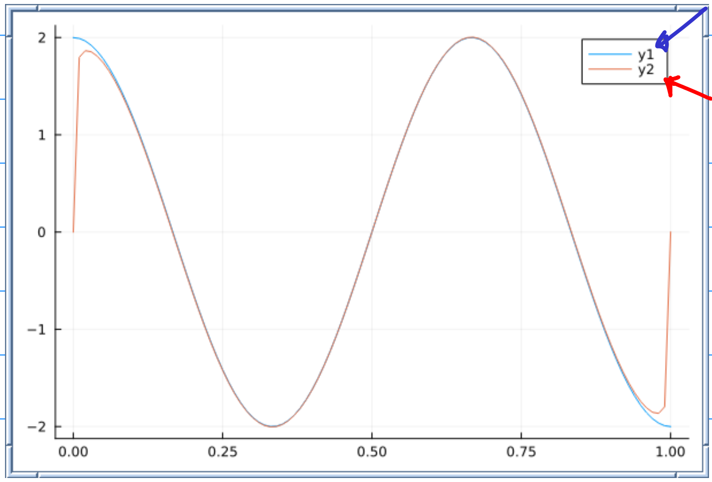


y_1 is desired initial cond

y_2 is the series approximation using 5 terms

Suggests the solution is correct. This can further be confirmed by summing 100 terms to obtain

.



y_1 is desired initial cond

y_2 is the series approximation using 100 terms

Note that, except for the discontinuity at the endpoints where the boundary condition forces the series solution to zero, that the series approximation is converging to the desired initial condition

***2.3.8.** Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u.$$

This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° ($\alpha > 0$, see Exercise 1.2.4) or with insulated lateral sides with a heat sink proportional to the temperature. Suppose that the boundary conditions are

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

(a) What are the possible equilibrium temperature distributions if $\alpha > 0$?

(b) Solve the time-dependent problem $[u(x,0) = f(x)]$ if $\alpha > 0$. Analyze the temperature for large time ($t \to \infty$) and compare to part (a).

(a) The system is in equilibrium is u does not depend on t in which case $\frac{\partial u}{\partial t} = 0$. Then

$$k\frac{d^2u}{dx^2} = \alpha u \qquad \text{or} \qquad \frac{d^2u}{dx^2} = \frac{\alpha}{k}u$$

which has a general solution

$$u(x) = Ae^{\sqrt{\alpha/k}\,x} + Be^{-\sqrt{\alpha/k}\,x}$$

or equivalently

$$u(x) = a\cosh\left(\sqrt{\frac{\alpha}{k}}\,x\right) + b\sinh\left(\sqrt{\frac{\alpha}{k}}\,x\right)$$

Solving for the boundary conditions as

$$u(0) = a\cosh(0) + b\sinh(0) = a = 0$$

and

$$u(L) = b\sinh\left(\sqrt{\frac{\alpha}{k}}L\right) = 0 \qquad \text{so } b = \frac{0}{\sin\left(\sqrt{\frac{\alpha}{k}}L\right)} = 0$$

implies the equilibrium solution is $u(x) = 0$.

To obtain the time dependent solution use separation of variables and the superposition principle. Thus

$$u(x,t) = \varphi(x) g(t)$$

and substituting yields

$$\varphi(x) g'(t) = k \varphi''(x) g(t) - \alpha \varphi(x) g(t)$$

$$\frac{g'(t)}{k g(t)} = \frac{\varphi''(x)}{\varphi(x)} - \frac{\alpha}{k}$$

and equivalently

$$\frac{g'(t)}{k g(t)} + \frac{\alpha}{k} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda .$$

Consequently

$$g'(t) + \alpha g(t) = -\lambda g(t) \qquad \text{and} \qquad \varphi''(x) = -\lambda \varphi(x)$$

Therefore

$$g(t) = C e^{-(\alpha+\lambda)t} \qquad \text{and} \qquad \varphi(x) = A \cos\sqrt{\lambda}\, x + B \sin\sqrt{\lambda}\, x$$

and after taking boundary conditions into account

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \qquad \text{and} \qquad \varphi(x) = B \sin\frac{n\pi x}{L}$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\alpha + \left(\frac{n\pi}{L}\right)^2\right)t} \sin\frac{n\pi x}{L}$$

where by orthogonality

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Now consider the limit $t \to \infty$.

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} \sum_{n=1}^{\infty} B_n e^{-\left(\alpha + \left(\frac{n\pi}{L}\right)^2\right)t} \sin\frac{n\pi x}{L}$$

Since

$$|B_n| = \left|\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right| \leq \frac{2}{L}\int_0^L |f(x)|\,dx = B < \infty$$

are bounded and for $t > 1$

$$e^{-\left(\alpha + \left(\frac{n\pi}{2}\right)^2\right)t} \leq e^{-\left(\alpha + \left(\frac{n\pi}{2}\right)^2\right)}$$

then

$$\sum_{n=1}^{\infty}\left|B_n e^{-\left(\alpha + \left(\frac{n\pi}{L}\right)^2\right)t} \sin\frac{n\pi x}{L}\right| \leq B\sum_{n=1}^{\infty} e^{-\left(\alpha + \left(\frac{n\pi}{2}\right)^2\right)} < \infty$$

shows the Fourier series is absolutely and uniformly convergent when $t \geq 1$. Thus, one can interchange the limits and obtain

$$\lim_{t\to\infty} u(x,t) = \sum_{n=1}^{\infty} B_n \lim_{t\to\infty} e^{-\left(\alpha + \left(\frac{n\pi}{L}\right)^2\right)t} \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} 0 = 0$$

which is the same as the equilibrium solution found earlier.

***2.4.2.** Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0$$
$$u(L,t) = 0$$
$$u(x,0) = f(x).$$

For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

For separation of variables set $u(x,t) = \phi(x)g(t)$ and plug this in to obtain

$$\phi(x)g'(t) = k\phi''(x)g(t)$$

or that

$$\frac{g'(t)}{kg(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda.$$

The homogeneous boundary conditions in x yield

$$\phi''(x) = -\lambda\phi(x) \quad \text{such that} \quad \phi'(0)=0 \text{ and } \phi(L)=0.$$

Case $\lambda>0$, The general solution is

$$\phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$
$$\phi'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

Therefore

$$\phi'(0) = -A\sqrt{\lambda}\sin(0) + B\sqrt{\lambda}\cos(0) = B\sqrt{\lambda}$$

implies $B=0$. Now

$$\phi(L) = A\cos\sqrt{\lambda}L = 0 \quad \text{shows} \quad \sqrt{\lambda}L = \frac{\pi}{2} + n\pi$$

Consequently,

$$\phi(x) = A\cos\left(\left(\frac{\pi}{2} + n\pi\right)\frac{x}{L}\right) \qquad \text{for } n = 0, 1, 2, \ldots$$

Case $\lambda = 0$. Then the general solution is

$$\phi(x) = Ax + B$$

and

$$\phi'(0) = A = 0 \qquad \text{followed by } \phi(L) = B = 0$$

imply there are no eigenfunctions when $\lambda = 0$. Further, we don't consider $\lambda < 0$ or else $g(t)$ below would grow exponentially in time.

Solving the differential equation for $g(t)$ yields

$$g(t) = Ce^{-k\lambda t} = Ce^{-k\left(\frac{\pi}{2} + n\pi\right)^2 t/L^2} \qquad \text{where } n = 0, 1, \ldots$$

By the superposition principle we obtain a more general solution that satisfies the boundary conditions

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-k\left(\frac{\pi}{2} + n\pi\right)^2 t/L^2} \cos\left(\left(\frac{\pi}{2} + n\pi\right)\frac{x}{L}\right)$$

where by orthogonality (assumed but could be proven) we have

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\left(\frac{\pi}{2} + n\pi\right)\frac{x}{L}\right)dx$$

Note that defining $c_n = A_{n-1}$ and shifting the index in the sums arrives at the answer in the back of the book. Namely,

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(n\pi - \frac{\pi}{2}\right)^2 t/L^2} \cos\left(\left(n\pi - \frac{\pi}{2}\right)\frac{x}{L}\right)$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\left(n\pi - \frac{\pi}{2}\right)\frac{x}{L}\right)dx.$$