HW1 problems 1.2.8, 1.3.2, 1.3.3, 1.4.1adfh, 1.4.2abc, 1.4.7a

***1.2.8.** If $u(x,t)$ is known, give an expression for the total thermal energy contained in a rod $(0 < x < L)$.

By definition the energy density $e = c\rho u$ where c is the heat capacity and ρ is the density. Thus

$$\text{Total energy} = \int_{\text{rod}} e = \int_0^L e\,A\,dx = A\int_0^L c\rho u\,dx$$

where A is the cross-sectional area of the rod.

***1.3.2.** Two one-dimensional rods of different materials joined at $x = x_0$ are said to be in **perfect thermal contact** if the temperature is continuous at $x = x_0$:

$$u(x_0-, t) = u(x_0+, t)$$

and no heat energy is lost at $x = x_0$ (i.e., the heat energy flowing out of one flows into the other). What mathematical equation represents the latter condition at $x = x_0$? Under what special condition is $\partial u/\partial x$ continuous at $x = x_0$?

No heat lost means the heat flux flowing through the surface at x_0- given by $\phi(x_0-,t)$ must equal the heat flux $\phi(x_0^+,t)$ at x_0^+ for all time.

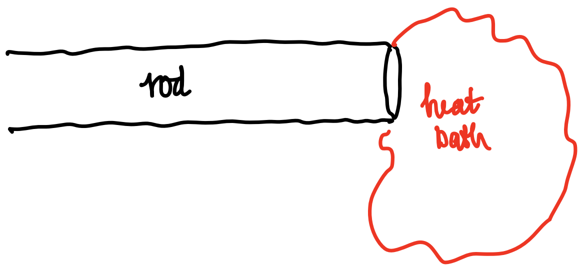
By Fourier's law the heat flux is given by $\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x}$ where $K_0(x)$ is the conductivity of the rod at any point x along the rod.

Thus, conservation of heat implies

$$K_0(x_0^-)\left.\frac{\partial u}{\partial t}\right|_{x=x_0^-} = K_0(x_0^+)\left.\frac{\partial u}{\partial t}\right|_{x=x_0^+}$$

Conclude $\frac{\partial u}{\partial t}$ is continuous when $K_0(x)$ is continuous. This means the conductivity of the two materials is the same.

***1.3.3.** Consider a bath containing a fluid of specific heat c_f and mass density ρ_f that surrounds the end $x = L$ of a one-dimensional rod. Suppose that the bath is rapidly stirred in a manner such that the bath temperature is approximately uniform throughout, equaling the temperature at $x = L, u(L, t)$. Assume that the bath is thermally insulated except at its perfect thermal contact with the rod, where the bath may be heated or cooled by the rod. Determine an equation for the temperature in the bath. (This will be a boundary condition at the end $x = L$.) (*Hint*: See Exercise 1.3.2.)



From the previous problem the heat flux leaving the rod is equal to the flux entering the heat bath.

The energy stored in the heat bath is

$$E_{bath} = c_f \rho_f V_{bath} u_{bath}$$

where u_{bath} is the temperature of the bath and V_{bath} is the volume of the bath. Since the bath is in perfect thermal contact with the rod, then $u_{bath} = u(L,t)$ for all time. Since no heat is lost, the rate of change in the energy stored in the bath is equal to the flux passing out of the end of the rod. By Fourier's law

$$\frac{dE_{bath}}{dt} = A\phi(L,t) = -K_0(L) A \frac{\partial u}{\partial x}\bigg|_{x=L}$$

where A is the cross-sectional area of the rod.

Since $\frac{dE_{bath}}{dt} = c_f \rho_f V_{bath} \frac{d}{dt} u_{bath} = c_f \rho_f V_{bath} \frac{\partial u}{\partial t}\bigg|_{x=L}$

it follows that

$$c_f \rho_f V_{bath} \frac{\partial u}{\partial t}\bigg|_{x=L} = -K_0(L) A \frac{\partial u}{\partial x}\bigg|_{x=L}.$$

1.4.1. Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

* **(a)** $Q = 0$, $u(0) = 0$, $u(L) = T$

Recall the heat equation with constant thermal properties given by

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial t^2} + Q$$

divide by K_0 to obtain

$$\frac{c\rho}{K_0} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} + \frac{Q}{K_0}$$

The equilibrium state $u(x)$ does not depend on time so $\frac{\partial u}{\partial t} = 0$. Setting $Q=0$ further yields

$$\frac{d^2 u}{dt^2} = 0 \quad \text{such that} \quad u(0)=0 \text{ and } u(L)=t.$$

The general solution to the ODE is $u(x) = mx+b$. Now solving for m and b in terms of the boundary conditions yields that

$$u(0) = b = 0 \quad \text{and} \quad u(L) = mL = T \quad \text{so } m = \frac{T}{L}$$

It follows that the equilibrium state is

$$u(x) = \frac{T}{L}x \quad \text{where} \quad x \in [0, L].$$

* **(d)** $Q = 0$, $u(0) = T$, $\frac{\partial u}{\partial x}(L) = \alpha$

This time $\frac{d^2u}{dt^2} = 0$ such that $u(0)=T$ and $\left.\frac{\partial u}{\partial x}\right|_{x=L} = \alpha$.

again the general solution is $u(x) = mx+b$. So

$$u(0) = b = T \qquad u'(x) = m \qquad \text{so} \quad u'(L) = m = \alpha$$

and the equilibrium state is $u(x) = \alpha x + T$.

$*$ **(f)** $\frac{Q}{K_0} = x^2, \qquad u(0) = T, \qquad \frac{\partial u}{\partial x}(L) = 0$

Since $\frac{d^2u}{dx^2} = -x^2$ the general solution is

$$u(x) = mx + b - \frac{1}{12}x^4$$

Solving for the constants as

$$u(0) = b = T$$

$$u'(x) = m - \frac{1}{3}x^3$$

$$u'(L) = m - \frac{1}{3}L^3 = 0 \quad \text{implies} \quad m = \frac{1}{3}L^3$$

Therefore the equilibrium solution is

$$u(x) = \frac{1}{3}L^3x + T - \frac{1}{12}x^4.$$

$*$ **(h)** $Q = 0, \qquad \frac{\partial u}{\partial x}(0) - [u(0) - T] = 0, \qquad \frac{\partial u}{\partial x}(L) = \alpha$

The ODE governing the equilibrium state is

$$\frac{d^2u}{dx^2} = 0 \quad \text{such that} \quad u'(0) = u(0) - T \text{ and } u'(L) = \alpha.$$

The general solution is

$$u(x) = mx + b$$

Solving for the constants as

$$u'(0) = m = u(0) - T$$

$$u'(L) = u(0) - T = \alpha \quad \text{so } u(0) = T + \alpha \text{ and } m = \alpha$$

and finally

$$u(0) = b = T + \alpha$$

we obtain the equilibrium state $u(x) = \alpha x + T + \alpha$.

1.4.2. Consider the equilibrium temperature distribution for a uniform one-dimensional rod with sources $Q/K_0 = x$ of thermal energy, subject to the boundary conditions $u(0) = 0$ and $u(L) = 0$.

*(a) Determine the heat energy generated per unit time inside the entire rod.

(b) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and at $x = L$.

(c) What relationships should exist between the answers in parts (a) and (b)?

The equilibrium temperature distribution is obtained by solving

$$\frac{d^2u}{dt^2} = -x \quad \text{such that } u(0)=0 \text{ and } u(L)=0$$

The general solution is $u(x) = mx + b - \frac{1}{6}x^3$. Solving for the parameters as

$$u(0) = b = 0$$

$$u(L) = mL - \frac{1}{6}L^3 = 0 \qquad m = \frac{1}{6}L^2$$

yields the equilibrium solution

$$u(x) = \frac{1}{6}L^2 x - \frac{1}{6}x^3.$$

(a) The rate of energy production inside the bar is

$$\int_0^L Q(x)\,dx = \int_0^L K_0 x\,dx = \frac{K_0}{2}x^2 \Big|_0^L = \frac{K_0}{2}L^2.$$

(b) The energy flux flowing out of the bar is

$$\phi(L) - \phi(0) = -K_0 u'(L) + K_0 u'(0).$$

Since $u'(x) = \frac{1}{6}L^2 - \frac{1}{2}x^2$ we obtain

$$\phi(L) - \phi(0) = -K_0\left(\cancel{\frac{1}{6}L^2} - \frac{1}{2}L^2 - \cancel{\frac{1}{6}L^2}\right) = K_0 \frac{1}{2}L^2.$$

(c) Conservation of energy states the energy produced must be equal to the energy flowing out for the bar to be in equilibrium. Thus the solution to (b) is equal the solution to (a).

1.4.7. For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

$*$ **(a)** $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1, \qquad u(x,0) = f(x), \qquad \dfrac{\partial u}{\partial x}(0,t) = 1, \qquad \dfrac{\partial u}{\partial x}(L,t) = \beta$

The equilibrium solution does not depend on t so $\frac{\partial u}{\partial t} = 0$. Thus we attempt to solve the ODE

$$\frac{d^2 u}{dx^2} = -1 \text{ such that } u'(0) = 1 \text{ and } u'(L) = \beta.$$

The general solution is $u(x) = mx + b - \frac{1}{2}x^2$.

Since $u'(x) = m - \frac{1}{2}x^2$ in order to satisfy the boundary conditions it must be that

$$u'(0) = m = 1 \qquad \text{so } m = 1$$
$$u'(L) = L - \tfrac{1}{2}L^2 = \beta \qquad \text{so } \beta = L - \tfrac{1}{2}L^2.$$

The only value of β for which there is an equilibrium solution is for $\beta = L - \frac{1}{2}L^2$.

Physically the energy produced within the bar and the flux leaving the bar must be the same for there to be an equilibrium solution. This uniquely determines β.

If $\beta < L - \frac{1}{2}L^2$ then more energy is being produced than leaving. That would cause the internal temperature of the bar to reach infinity, which is non-physical.

If $\beta > L - \frac{1}{2}L^2$ then more energy is leaving than being produced. Over time this would cause the temperature to go to negative infinity, which is again non-physical.