

HW4 problems 4.2.2, 4.4.1, 4.4.2, 4.4.5, 4.4.10, 5.3.1

4.2.2. Show that c^2 has the dimensions of velocity squared.

The wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The dimensions of $\frac{\partial^2 u}{\partial t^2}$ are

$$\left[\frac{\partial^2 u}{\partial t^2} \right] = \frac{[u]}{[T]^2}$$

The dimensions of $\frac{\partial^2 u}{\partial x^2}$ are

$$\left[\frac{\partial^2 u}{\partial x^2} \right] = \frac{[u]}{[L]^2}$$

Therefore, the dimensions of c^2 are

$$[c^2] = \frac{\left[\frac{\partial^2 u}{\partial t^2} \right]}{\left[\frac{\partial^2 u}{\partial x^2} \right]} = \frac{\frac{[u]}{[T]^2}}{\frac{[u]}{[L]^2}} = \frac{[L]^2}{[T]^2}$$

It follows that the dimensions of c are

$$[c] = \frac{[L]}{[T]}$$

which is a velocity.

4.4.1. Consider vibrating strings of uniform density ρ_0 and tension T_0 .

- *(a) What are the natural frequencies of a vibrating string of length L fixed at both ends?
- *(b) What are the natural frequencies of a vibrating string of length H , which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u / \partial x(H, t) = 0$]? Sketch a few modes of vibration as in Fig. 4.4.1.
- (c) Show that the modes of vibration for the *odd* harmonics (i.e., $n = 1, 3, 5, \dots$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

(a) Consider $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T_0}{\rho_0}$.

If the boundary conditions are fixed as

$$u(0, t) = a \quad \text{and} \quad u(L, t) = b$$

then consider

$$w(x, t) = u(x, t) - a - \frac{(b-a)}{L} x.$$

Since $\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$ it follows that w satisfies

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

with the homogeneous boundary conditions

$$w(0, t) = 0 \quad \text{and} \quad w(L, t) = 0$$

Performing separation of variables

$$w(x, t) = \phi(x) h(t)$$

yields

$$\varphi(x) h''(t) = c^2 \varphi''(x) h(t)$$

so that

$$\frac{h''(t)}{c^2 h(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

Therefore

$$h''(t) = -\lambda c^2 h(t) \quad \text{and} \quad \varphi''(x) = -\lambda \varphi(x).$$

In general

$$\varphi(x) = a \cos \sqrt{\lambda} x + b \sin \sqrt{\lambda} x$$

while the boundary conditions

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(L) = 0$$

imply

$$\varphi(0) = a \cos \sqrt{\lambda} 0 + b \sin \sqrt{\lambda} 0 = a = 0$$

$$\text{and} \quad \varphi(L) = b \sin \sqrt{\lambda} L = 0.$$

since $b \neq 0$ then $\sin \sqrt{\lambda} L = 0$ or $\sqrt{\lambda} L = n\pi$.

Now

$$\begin{aligned} h(t) &= \alpha \cos(c\sqrt{\lambda} t) + \beta \sin(c\sqrt{\lambda} t) \\ &= \alpha \cos \frac{cn\pi t}{L} + \beta \sin \frac{cn\pi t}{L}. \end{aligned}$$

By superposition

$$w(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{cn\pi t}{L} + b_n \sin \frac{cn\pi t}{L} \right) \cos \frac{n\pi x}{L}$$

The natural frequencies can be seen in the time dependent part of the above expression. Namely, the frequencies are

$$\frac{cn\pi}{L} \quad \text{for } n=1,2,3,\dots$$

(b) If one end of the string is free, then we have the boundary conditions

$$f(0) = 0 \quad \text{and} \quad f'(L) = 0$$

again taking

$$f(x) = a \cos \sqrt{\lambda} x + b \sin \sqrt{\lambda} x$$

yields

$$f(0) = a = 0$$

$$f'(x) = b\sqrt{\lambda} \cos \sqrt{\lambda} x$$

$$f'(L) = b\sqrt{\lambda} \cos \sqrt{\lambda} L = 0$$

since $b \neq 0$ then $\sqrt{\lambda} L = (n + \frac{1}{2})\pi$ for $n = 0, 1, 2, \dots$

By superposition

$$w(x,t) = \sum_{n=0}^{\infty} \left(a_n \cos \frac{c(n+\frac{1}{2})\pi t}{H} + b_n \sin \frac{c(n+\frac{1}{2})\pi t}{H} \right) \sin \frac{(n+\frac{1}{2})\pi x}{H}$$

In this case the natural frequencies are

$$\frac{c(n+\frac{1}{2})\pi}{H} \quad \text{for } n=0, 1, 2, \dots$$

(c) Taking $H = L/2$ in part (b) yields the modes

$$\sin \frac{(n+\frac{1}{2})\pi x}{H} = \sin \frac{(n+\frac{1}{2})\pi x}{L/2} = \sin \frac{(2n+1)\pi x}{L}.$$

for $n=0, 1, 2, \dots$,

Taking $n=2k+1$ in part (a) yields the modes

$$\sin \frac{n\pi x}{L} = \sin \frac{(2k+1)\pi x}{L} \quad \text{for } k=0, 1, 2, \dots$$

These are both the same collection of modes.

4.4.2. In Section 4.2 it was shown that the displacement u of a nonuniform string satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + Q,$$

where Q represents the vertical component of the body force per unit length. If $Q = 0$, the partial differential equation is homogeneous. A slightly different homogeneous equation occurs if $Q = \alpha u$.

- (a) Show that if $\alpha < 0$, the body force is restoring (toward $u = 0$). Show that if $\alpha > 0$, the body force tends to push the string further away from its unperturbed position $u = 0$.
- (b) Separate variables if $\rho_0(x)$ and $\alpha(x)$ but T_0 is constant for physical reasons. Analyze the time-dependent ordinary differential equation.
- * (c) Specialize part (b) to the constant coefficient case. Solve the initial value problem if $\alpha < 0$:

$$u(0, t) = 0, \quad u(x, 0) = 0$$

$$u(L, t) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = f(x).$$

What are the frequencies of vibration?

(a) Suppose $\alpha < 0$ then $\alpha = -|\alpha|$ and

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - |\alpha| u$$

Since the force term $-|\alpha|u$ is negative when $u > 0$ that tends to push u down. On the other hand $u < 0$ leads to a positive force which tends to push u upwards.

Suppose $\alpha > 0$ then $\alpha = |\alpha|$ and

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + |\alpha| u$$

In this case $u > 0$ results in a positive force that pushes u further up and $u < 0$ results in a negative force that will push u further down.

(b) We have

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha(x) u$$

For separation of variables let

$$u(x,t) = q(x)h(t).$$

Then

$$\rho_0(x) q(x) h''(t) = T_0 q''(x) h(t) + \alpha(x) q(x) h(t)$$

Dividing by $h(t)$ yields

$$\rho_0(x) q(x) \frac{h''(t)}{h(t)} = T_0 q''(x) + \alpha(x) q(x)$$

and then

$$\frac{h''(t)}{h(t)} = \frac{T_0 q''(x) + \alpha(x) q(x)}{\rho_0(x) q(x)} = -\lambda$$

Consequently

$$h''(t) = -\lambda h(t) \quad \text{and} \quad T_0 q''(x) = (-\lambda \rho_0(x) - \alpha(x)) q(x)$$

Analysis of the time dependent consists of three cases.

Case $\lambda < 0$ then $-\lambda = |\lambda|$ and

$$h(t) = a e^{-\sqrt{|\lambda|}t} + b e^{\sqrt{|\lambda|}t}$$

or equivalently

$$h(t) = a \cosh \sqrt{\lambda} t + b \sinh \sqrt{\lambda} t$$

In this case the time dependent behavior is exponential.

Case $\lambda = 0$ then

$$h(t) = at + b$$

and the time behavior is linear.

Case $\lambda > 0$ then $-\lambda = -|\lambda|$

$$h(t) = a \cos \sqrt{\lambda} t + b \sin \sqrt{\lambda} t$$

and the time behavior is oscillatory.

(c) The constant coefficient case with $\alpha < 0$ is

$$\frac{h''(t)}{h(t)} = \frac{T_0 q''(x) + \alpha q(x)}{\rho_0 q(x)} = -\lambda$$

So that

$$q''(x) = \frac{|\alpha| - \lambda \rho_0}{T_0} q(x)$$

subject to the boundary conditions

$$q(0) = 0 \quad \text{and} \quad q(L) = 0.$$

If $\frac{|\alpha| - \lambda \rho_0}{T_0} \geq 0$ it is known that there are no non-zero solutions which satisfy the boundary conditions. Thus, we assume $\lambda \rho_0 > |\alpha|$.

In this case the general solution is

$$\varphi(x) = a \cos\left(\sqrt{\frac{\lambda \rho_0 - |\alpha|}{T_0}} x\right) + b \sin\left(\sqrt{\frac{\lambda \rho_0 - |\alpha|}{T_0}} x\right)$$

The boundary conditions then imply

$$\varphi(0) = a = 0$$

$$\varphi(L) = b \sin\left(\sqrt{\frac{\lambda \rho_0 - |\alpha|}{T_0}} L\right) = 0$$

and so $\sqrt{\frac{\lambda \rho_0 - |\alpha|}{T_0}} L = n\pi$ for $n = 1, 2, 3, \dots$

It follows that

$$\frac{\lambda \rho_0 - |\alpha|}{T_0} = \left(\frac{n\pi}{L}\right)^2, \quad \lambda \rho_0 = |\alpha| + T_0 \left(\frac{n\pi}{L}\right)^2$$

and finally $\lambda = \frac{|\alpha| + T_0 \left(\frac{n\pi}{L}\right)^2}{\rho_0} > 0$

for $n = 1, 2, 3, \dots$

Therefore, the time dependent part is oscillatory and by the superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos \left(\sqrt{\frac{|d| + T_0 \left(\frac{n\pi}{L} \right)^2}{\rho_0}} t \right) + b_n \sin \left(\sqrt{\frac{|d| + T_0 \left(\frac{n\pi}{L} \right)^2}{\rho_0}} t \right) \right] \sin \frac{n\pi x}{L}$$

Now solve for a_n and b_n using the initial conditions,

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} = 0$$

implies $a_n = 0$ for all $n = 1, 2, \dots$. Thus

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \left(\sqrt{\frac{|d| + T_0 \left(\frac{n\pi}{L} \right)^2}{\rho_0}} t \right) \sin \frac{n\pi x}{L}$$

and differentiating yields

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \sqrt{\frac{|d| + T_0 \left(\frac{n\pi}{L} \right)^2}{\rho_0}} \sin \frac{n\pi x}{L} = f(x).$$

It follows that

$$b_n \sqrt{\frac{|d| + T_0 \left(\frac{n\pi}{L} \right)^2}{\rho_0}} = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

or equivalently

$$b_n = \sqrt{\frac{\rho_0}{|d| + T_0 \left(\frac{n\pi}{L} \right)^2}} \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

4.4.5. Redo Exercise 4.4.3(b) if $4\pi^2\rho_0\tilde{T}_0/L^2 < \beta^2 < 16\pi^2\rho_0T_0/L^2$.

Recall 4.4.3 is

4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

- (a) Briefly explain why $\beta > 0$.
 *(b) Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0$$

and the initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2\rho_0T_0/L^2$).

(a) The total energy in the string is given by

$$E = \frac{1}{2} \int_0^L \left(\frac{\partial u}{\partial t} \right)^2 dx + \frac{c^2}{2} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \quad \text{where } c^2 = \frac{T_0}{\rho_0}$$

Therefore

$$\frac{\partial E}{\partial t} = \frac{1}{2} \int_0^L \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right)^2 dx + \frac{c^2}{2} \int_0^L \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right)^2 dx$$

$$= \int_0^L \left(\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} \right) dx$$

$$= \int_0^L \left[\frac{\partial u}{\partial t} \left(c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\beta}{T_0} \frac{\partial u}{\partial t} \right) + c^2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} \right] dx$$

Now since

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right) = \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t}$$

then

$$\frac{\partial E}{\partial t} = \int_0^L \left[c^2 \left(\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right) - \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} \right) - \frac{\beta}{T_0} \left(\frac{\partial u}{\partial t} \right)^2 + c^2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} \right] dx$$

$$= c^2 \left. \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right|_0^L - \frac{\beta}{T_0} \int_0^L \left(\frac{\partial u}{\partial t} \right)^2 dx$$

Assuming the boundary terms vanish as they do for part (b) then we have

$$\frac{\partial E}{\partial t} = - \frac{\beta}{T_0} \int_0^L \left(\frac{\partial u}{\partial t} \right)^2 dx$$

This shows that if $\beta > 0$ then $\frac{\partial E}{\partial t} < 0$ and so the energy is decreasing over time. This is damping.

On the other hand, if $\beta < 0$ then $\frac{\partial E}{\partial t} > 0$ and so the energy is increasing over time, which is not damping.

If $\beta = 0$ the term vanishes and the system conserves energy.

A brief explanation appears in lecture 15 in the notes.

(b) For separation of variables

$$u(x,t) = \phi(x) h(t)$$

Then

$$\rho_0 \phi(x) h''(t) = T_0 \phi''(x) h(t) - \beta h'(t)$$

$$\frac{\rho_0}{T_0} \frac{h''(t)}{h(t)} = \frac{\phi''(x)}{\phi(x)} - \frac{\beta}{T_0} \frac{h'(t)}{h(t)}$$

thus,

$$\frac{\rho_0}{T_0} \frac{h''(t)}{h(t)} + \frac{\beta}{T_0} \frac{h'(t)}{h(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

we obtain

$$\rho_0 h''(t) + \beta h'(t) = -\lambda T_0 h(t) \quad \text{and} \quad \phi''(x) = -\lambda \phi(x)$$

The general solution for $\phi(x)$ is

$$\phi(x) = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x$$

The boundary conditions $\phi(0) = 0$ and $\phi(L) = 0$ then imply

$$\phi(0) = A = 0$$

$$\phi(L) = B \sin \sqrt{\lambda} L = 0 \quad \text{so} \quad \sqrt{\lambda} L = n\pi$$

$$\text{or} \quad \sqrt{\lambda} = \frac{n\pi}{L} \quad \text{for } n=1, 2, \dots$$

Now solve for $h(t)$ by plugging in e^{rt}

$$\rho_0 r^2 e^{rt} + \beta r e^{rt} = -\lambda T_0 e^{rt}$$

$$\text{implies } \rho_0 r^2 + \beta r + \lambda T_0 = 0.$$

By the quadratic formula with $a = \rho_0$, $b = \beta$ and $c = \lambda T_0$ we obtain that

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \lambda T_0}}{2\rho_0} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0}}{2\rho_0}$$

The condition $4\pi^2 \rho_0 T_0 / L^2 < \beta^2 < 16\pi^2 \rho_0 T_0 / L^2$ implies

$$\beta^2 - 4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 > 0 \quad \text{for } n=1$$

$$\beta^2 - 4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 < 0 \quad \text{for } n=2, 3, 4, \dots$$

Therefore, break the solution into two cases.

Case $n=1$

$$h(t) = a_1 \cosh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0 \left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right) + b_1 \sinh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0 \left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right)$$

Case $n=2$

$$h(t) = e^{-\frac{\beta}{2\rho_0} t} \left[a_n \cos\left(\frac{\sqrt{4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) + b_n \sin\left(\frac{\sqrt{4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) \right]$$

By superposition it follows that

$$u(x,t) =$$

$$\left[a_1 \cosh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right) + b_1 \sinh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right) \right] \sin \frac{\pi x}{L}$$

$$+ \sum_{n=2}^{\infty} e^{\frac{-\beta}{2\rho_0} t} \left[a_n \cos\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) + b_n \sin\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) \right] \sin \frac{n\pi x}{L}$$

Solving for the initial conditions yields

$$u(x,0) = \sum a_n \sin \frac{n\pi x}{L} \approx f(x)$$

$$\text{and so } a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Then

$$\frac{\partial u}{\partial t}(x,t) =$$

$$\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0}\right) \left[a_1 \sinh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right) + b_1 \cosh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right) \right] \sin \frac{\pi x}{L}$$

$$+ \sum_{n=2}^{\infty} \left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0}\right) e^{\frac{-\beta}{2\rho_0} t} \left[-a_n \sin\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) + b_n \cos\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) \right] \sin \frac{n\pi x}{L}$$

$$+ \sum_{n=2}^{\infty} \frac{-\beta}{2\rho_0} e^{\frac{-\beta}{2\rho_0} t} \left[a_n \cos\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) + b_n \sin\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) \right] \sin \frac{n\pi x}{L}$$

Consequently

$$\frac{\partial u}{\partial t}(x,0) = \left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0 \left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} \right) b_1 \sin \frac{\pi x}{L} + \sum_{n=2}^{\infty} \left[\frac{\sqrt{4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} \right] \left(b_n - \frac{\beta}{2\rho_0} a_n \right) \sin \frac{n\pi x}{L}$$

and it follows that

$$b_1 = \frac{2\rho_0}{-\beta + \sqrt{\beta^2 - 4\rho_0 \left(\frac{\pi}{L}\right)^2 T_0}} \frac{2}{L} \int_0^h g(x) \sin \frac{\pi x}{L} dx$$

and for $n \geq 2$ that

$$b_n = \frac{2\rho_0}{\sqrt{4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}} \left(\frac{\beta}{2\rho_0} a_n + \frac{2}{L} \int_0^h g(x) \sin \frac{n\pi x}{L} dx \right).$$

4.4.10. What happens to the total energy E of a vibrating string (see Exercise 4.4.9)

(a) If $u(0, t) = 0$ and $u(L, t) = 0$?

(b) If $\frac{\partial u}{\partial x}(0, t) = 0$ and $u(L, t) = 0$?

(c) If $u(0, t) = 0$ and $\frac{\partial u}{\partial x}(L, t) = -\gamma u(L, t)$ with $\gamma > 0$?

(d) If $\gamma < 0$ in part (c)?

Recall from Exercise 4.4.9 that

$$\frac{dE}{dt} = c^2 \left. \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right|_0^L$$

(a) Since $\frac{\partial u}{\partial t}(0, t) = \frac{\partial 0}{\partial t} = 0$ and $\frac{\partial u}{\partial t}(L, t) = \frac{\partial 0}{\partial t} = 0$ then

$$\frac{dE}{dt} = c^2 \left[\frac{\partial u}{\partial x}(L, t) \frac{\partial u}{\partial t}(L, t) - \frac{\partial u}{\partial x}(0, t) \frac{\partial u}{\partial t}(0, t) \right] = 0.$$

Consequently the energy is conserved.

(b) In this case

$$\frac{dE}{dt} = c^2 \left[\frac{\partial u}{\partial x}(L, t) \frac{\partial u}{\partial t}(L, t) - \frac{\partial u}{\partial x}(0, t) \frac{\partial u}{\partial t}(0, t) \right] = 0.$$

Consequently the energy is conserved.

$$\begin{aligned} \text{(c)} \quad \frac{dE}{dt} &= c^2 \left[\frac{\partial u}{\partial x}(L, t) \frac{\partial u}{\partial t}(L, t) - \frac{\partial u}{\partial x}(0, t) \frac{\partial u}{\partial t}(0, t) \right] \\ &= c^2 \left[-\gamma u(L, t) \frac{\partial u}{\partial t}(L, t) \right] = \frac{-\gamma c^2}{2} \frac{d}{dt} u^2(L, t) \end{aligned}$$

Integration then yields that

$$E(t) - E(0) = -\frac{\gamma c^2}{2} [u^2(L,t) - u^2(L,0)]$$

or

$$E(t) = E(0) - \frac{\gamma c^2}{2} [u^2(L,t) - u^2(L,0)].$$

Since $\gamma > 0$ then $u^2(L,t) > u^2(L,0)$ implies this term decreases the total energy while $u^2(L,t) < u^2(L,0)$ leads to an increase. I think the effect will stabilize the amount of energy in the system but am not sure.

(d) When $\gamma < 0$ then $u^2(L,t) > u^2(L,0)$ implies this term increases the total energy. This may cause total energy to increase.

*5.3.1. Do Exercise 4.4.2(b). Show that the partial differential equation may be put into Sturm–Liouville form.

The equation in Exercise 4.4.2 part (b) is

$$T_0 \phi''(x) = (-\lambda \rho_0(x) - \alpha(x)) \phi(x)$$

We now seek to identify p , q and σ with $p > 0$ and $\sigma > 0$ that rewrite this equation as

$$\frac{d}{dx} \left(p \frac{d\phi}{dx} \right) + q\phi + \lambda\sigma\phi = 0,$$

Since T_0 does not depend on x , then we can move it through the first derivative and rearrange the terms to obtain

$$\frac{d}{dx} (T_0 \phi'(x)) + \alpha(x) \phi(x) + \lambda \rho_0(x) \phi(x) = 0$$

From this it is clear that

$$p(x) = T_0, \quad q(x) = \alpha(x) \quad \text{and} \quad \sigma(x) = \rho_0(x).$$

Recalling that $T_0 > 0$ and the density $\rho_0(x) > 0$ finishes the identification.