***5.3.3.** Consider the non-Sturm–Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm–Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

Given $\alpha(x), \beta(x)$, and $\gamma(x)$, what are $p(x), \sigma(x)$, and $q(x)$?

Multiplying yields

$$H\frac{d^2\phi}{dx^2} + H\alpha\frac{d\phi}{dx} + (H\lambda\beta + H\gamma)\phi = 0.$$

To make the second term go away we need

$$\frac{dH\phi'}{dx} = H\frac{d^2\phi}{dx^2} + H\alpha\frac{d\phi}{dx}$$

Since

$$\frac{dH\phi'}{dx} = H'\phi' + H\frac{d^2\phi}{dx^2}$$

this implies $H' = H\alpha$. Consequently

$$\int_{H_0}^{H(x)} \frac{dH}{H} = \int_0^x \alpha(x)\,dx$$

$$\log H(x) - \log H_0 = \int_0^x \alpha(x)\,dx$$

or

$$H(x) = H_0\, e^{\int_0^x \alpha(x)\,dx}$$

With this choice of H we obtain

$$\frac{d\,H y'}{dx} + (H\lambda\beta + H\gamma)\,y = 0.$$

which is the standard Sturm-Liouville form where

$$p(x) = H(x) = H_0\, e^{\int_0^x \alpha(x)\,dx}$$

$$r(x) = H(x)\beta(x) = H_0\,\beta(x)\, e^{\int_0^x \alpha(x)\,dx}$$

and

$$q(x) = H(x)\,\gamma(x) = H_0\,\gamma(x)\, e^{\int_0^x \alpha(x)\,dx}.$$

5.3.4. Consider heat flow with convection (see Exercise 1.5.2):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - V_0\frac{\partial u}{\partial x}.$$

(a) Show that the spatial ordinary differential equation obtained by separation of variables is not in Sturm–Liouville form.

For separation of variables write

$$u(x,t) = \phi(x)h(t)$$

Then

$$\phi(x)h'(t) = k\phi''(x)h(t) - V_0\phi'(x)h(t)$$

and

$$\frac{h'(t)}{h(t)} = \frac{k\phi''(x) - V_0\phi'(x)}{\phi(x)} = -\lambda$$

Consequently,

$$h'(t) = -\lambda h(t) \quad \text{and} \quad \phi''(x) - \frac{V_0}{k}\phi'(x) + \frac{\lambda}{k}\phi(x) = 0$$

The equation in ϕ'' is not in Sturm-Liouville form.

Note that this equation can be put in Sturm-Liouville form by multiplying it by

$$H(x) = H_0 e^{-\int_0^x \frac{V_0}{k}dx} = H_0 e^{-V_0 x/k}.$$

Thus $H'(x) = \frac{-V_0}{k}H(x)$ implies

$$\frac{d}{dx}(H(x)\phi'(x)) + \frac{\lambda}{K}H(x)\phi(x) = 0$$

which is in Sturm-Liouville form with

$$p(x) = H(x) = H_0 e^{-V_0 x/K}$$

and $$\sigma(x) = \frac{H(x)}{K} = \frac{H_0}{K} e^{-V_0 x/K}.$$

*(b) Solve the initial boundary value problem

$$u(0,t) = 0$$
$$u(L,t) = 0$$
$$u(x,0) = f(x).$$

It's easier to solve the original ODE directly. So consider

$$K\phi''(x) - V_0\phi'(x) + \lambda\phi(x) = 0 \quad \text{with } \phi(0)=0 \text{ and } \phi(L)=0$$

Substituting e^{rx} yields

$$Kr^2 - V_0 r + \lambda = 0$$

consequently, by the quadratic formula

$$r = \frac{V_0 \pm \sqrt{V_0^2 - 4K\lambda}}{2K}$$

If $v_0^2 > 4k\lambda$ then the solution is given by exponentials in which case

$$f(x) = Ae^{r_1 x} + Be^{r_2 x}$$

and it is impossible for a non-zero solution to satisfy the boundary conditions.

Similarly if $v_0^2 = 4k\lambda$ no non-zero solution satisfies the boundary conditions.

Therefore, $v_0^2 < 4k\lambda$ and the solution is

$$f(x) = e^{\frac{v_0}{2k}x}\left(A\cos\frac{\sqrt{4k\lambda - v_0^2}}{2k}x + B\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x\right)$$

Since

$$f(0) = A = 0$$

and

$$f(L) = e^{\frac{v_0}{2k}L}\, B\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}L = 0$$

it follows that $\frac{\sqrt{4k\lambda - v_0^2}}{2k}L = n\pi$ for $n = 1, 2, \ldots$

Consequently $4k\lambda - v_0^2 = \left(\frac{2kn\pi}{L}\right)^2$

or $\lambda = k\left(\frac{n\pi}{L}\right)^2 + \frac{v_0^2}{4k}$

Since solving for $h(t)$ yields that

$$h(t) = Ce^{-\lambda t}$$

the superposition principle implies

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t} e^{\frac{V_0}{2k}x} \sin\frac{n\pi x}{L}$$

where $\lambda_n = k\left(\frac{n\pi}{L}\right)^2 + \frac{V_0^2}{4k}$.

Since the ODE for ϕ can be put into Sturm-Liouville form we know that the eigenfunctions

$$e^{\frac{V_0}{2k}x} \sin\frac{n\pi x}{L}$$

are orthogonal relative the weight function

$$\sigma(x) = e^{-V_0 x/k}$$

where we have taken $H_0 = k$ for convenience.

Solving for the constants B_n to satisfy the initial conditions using orthogonality then yields

$$u(x,0) = \sum_{n=1}^{\infty} B_n e^{\frac{V_0}{2k}x} \sin\frac{n\pi x}{L} = f(x)$$

so that

$$\sum_{n=1}^{\infty} B_n e^{\frac{V_0}{2k}x} \sin\frac{n\pi x}{L} e^{\frac{V_0}{2k}x} \sin\frac{m\pi x}{L} e^{-\frac{V_0}{k}x} = f(x) e^{\frac{V_0}{2k}x} \sin\frac{m\pi x}{L} e^{-\frac{V_0}{k}x}$$

or

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} = f(x)\, e^{-\frac{v_0}{2k}x} \sin\frac{m\pi x}{L} .$$

Integrating then yields

$$B_m = \frac{2}{h}\int_0^L f(x)\, e^{-\frac{v_0}{2k}x} \sin\frac{m\pi x}{L}\, dx$$

Note carefully the sign on the argument to the exponential comes from using the weight needed for orthogonality.

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t} e^{\frac{v_0}{2k}x} \sin\frac{n\pi x}{L}$$

where $\lambda_n = k\left(\frac{n\pi}{L}\right)^2 + \frac{v_0^2}{4k}$

and $B_n = \frac{2}{h}\int_0^L f(x)\, e^{-\frac{v_0}{2k}x} \sin\frac{n\pi x}{L}\, dx .$

(c) Solve the initial boundary value problem

$$\frac{\partial u}{\partial x}(0,t) = 0$$
$$\frac{\partial u}{\partial x}(L,t) = 0$$
$$u(x,0) = f(x).$$

Begin by solving

$$k\varphi''(x) - V_0\varphi'(x) + \lambda\varphi(x) = 0 \quad \text{with } \varphi'(0)=0 \text{ and } \varphi'(L)=0.$$

Again substituting e^{rx} yields

$$r = \frac{V_0 \pm \sqrt{V_0^2 - 4k\lambda}}{2k}$$

In the case $V_0^2 = 4k\lambda$ we obtain

$$\varphi(x) = Ae^{\frac{V_0}{2k}x} + Bxe^{\frac{V_0}{2k}x}$$

Now $\varphi'(x) = \left(\frac{V_0}{2k}A + B\right)e^{\frac{V_0}{2k}x} + \frac{V_0}{2k}Bx\,e^{\frac{V_0}{2k}x}$

$$\varphi'(0) = \frac{V_0}{2k}A + B = 0 \qquad \text{implies} \quad B = -\frac{V_0}{2k}A$$

Then

$$\varphi'(L) = \frac{V_0}{2k}BL\,e^{\frac{V_0}{2k}L} = 0 \qquad \text{implies} \quad A = B = 0$$

Therefore, the only eigenfunctions are again when $V_0^2 < 4k\lambda$

$$\varphi(x) = e^{\frac{V_0}{2k}x}\left(A\cos\frac{\sqrt{4k\lambda - V_0^2}}{2k}x + B\sin\frac{\sqrt{4k\lambda - V_0^2}}{2k}x\right).$$

Differentiating yields

$$\varphi'(x) = \frac{v_0}{2k} e^{\frac{v_0}{2k}x}\left(A\cos\frac{\sqrt{4k\lambda - v_0^2}}{2k}x + B\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x\right)$$

$$+ \frac{\sqrt{4k\lambda - v_0^2}}{2k} e^{\frac{v_0}{2k}x}\left(-A\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x + B\cos\frac{\sqrt{4k\lambda - v_0^2}}{2k}x\right).$$

Thus

$$\varphi'(0) = \frac{v_0}{2k}A + \frac{\sqrt{4k\lambda - v_0^2}}{2k}B = 0 \qquad \text{so} \quad A = -\frac{\sqrt{4k\lambda - v_0^2}}{v_0}B$$

consequently

$$\varphi'(x) = \frac{v_0}{2k} e^{\frac{v_0}{2k}x} B\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x + \frac{4k\lambda - v_0^2}{2kv_0} e^{\frac{v_0}{2k}x} B\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x$$

$$= Be^{\frac{v_0}{2k}x}\left(\frac{v_0}{2k} + \frac{4k\lambda - v_0^2}{2kv_0}\right)\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x$$

and

$$\varphi'(L) = Be^{\frac{v_0}{2k}x}\left(\frac{v_0}{2k} + \frac{4k\lambda - v_0^2}{2kv_0}\right)\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}L = 0$$

implies $\frac{\sqrt{4k\lambda - v_0^2}}{2k}L = n\pi$

or $\quad \lambda = k\left(\frac{n\pi}{L}\right)^2 + \frac{v_0^2}{4k} \quad$ for $\; n = 0, 1, 2, \ldots$

Note that $n=0$ now corresponds to a non-zero eigenfunction, the constant eigenfunction, so we keep it.

Therefore

$$A = -\frac{\sqrt{4k\lambda - v_0^2}}{v_0} B = -\frac{2k}{v_0} \cdot \frac{n\pi}{L} B$$

and

$$f(x) = B e^{\frac{v_0}{2k}x} \left(-\frac{2k}{v_0} \cdot \frac{n\pi}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi x}{L} \right).$$

Now we see that $n=0$ doesn't work. So by superposition

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t} e^{\frac{v_0}{2k}x} \left(-\frac{2k}{v_0} \cdot \frac{n\pi}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi x}{L} \right)$$

where $\lambda_n = k\left(\frac{n\pi}{L}\right)^2 + \frac{v_0^2}{4k}$.

Now use orthogonality with respect to the weight $e^{-\frac{v_0 x}{k}}$ to solve for the initial condition

$$u(x,0) = \sum_{n=1}^{\infty} B_n e^{\frac{v_0}{2k}x} \left(-\frac{2k}{v_0} \cdot \frac{n\pi}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi x}{L} \right) = f(x)$$

Orthogonality implies

$$B_n \left(-\frac{k n\pi}{v_0} + \frac{L}{2} \right) = \int_0^L f(x) e^{-\frac{v_0}{2k}x} \left(-\frac{2k}{v_0} \cdot \frac{n\pi}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi x}{L} \right) dx$$

Therefore the solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t} e^{\frac{v_0}{2k}x}\left(-\frac{2k}{v_0}\cdot\frac{n\pi}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi x}{L}\right)$$

where $\lambda_n = k\left(\frac{n\pi}{L}\right)^2 + \frac{v_0^2}{4k}$ and

$$B_n = \frac{1}{-\frac{kn\pi}{v} + \frac{L}{2}} \int_0^L f(x)\, e^{-\frac{v_0}{2k}x}\left(-\frac{2k}{v_0}\cdot\frac{n\pi}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi x}{L}\right)dx.$$

5.3.8. Show that $\lambda \geq 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(1) = 0.$$

Is $\lambda = 0$ an eigenvalue?

This is a Sturm-Liouville problem with

$p(x) = 1$, $q(x) = -x^2$ and $\sigma(x) = 1$.

The Rayleigh quotient is

$$\lambda = \frac{\phi(x)\phi'(x)\Big|_0^1 + \int_0^1 \left(\phi'(x)^2 + x^2 \phi(x)^2\right) dx}{\int_0^1 \phi(x)^2\, dx}$$

$$= \frac{\int_0^1 \left(\phi'(x)^2 + x^2 \phi(x)^2\right) dx}{\int_0^1 \phi(x)^2\, dx}$$

Since $\phi(x) \neq 0$ then $\int_0^1 x^2 \phi(x)^2\, dx > 0$. It follows that

$$\int_0^1 \left(\phi'(x)^2 + x^2 \phi(x)^2\right) dx > 0$$

and consequently that $\lambda > 0$.

Therefore $\lambda = 0$ is not an eigenvalue.

*12.2.2. Solve

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0 \qquad \text{with } w(x,0) = \cos x.$$

Put $x = x(t)$ and consider

$$\frac{d}{dt} w(x(t), t) = \frac{\partial w}{\partial x} x'(t) + \frac{\partial w}{\partial t}.$$

If $x'(t) = -3$ then $\frac{d}{dt} w(x(t), t) = 0$.

Solving these two ordinary differential equations then gives

$$x(t) = -3t + x_0 \qquad \text{and} \qquad w(x(t), t) = w(x_0, 0) = \cos x_0.$$

It follows upon substitution that

$$w(-3t + x_0, t) = \cos x_0$$

Solving for x_0 in terms of x as

$$x = -3t + x_0 \qquad \text{or} \qquad x_0 = x + 3t$$

then yields

$$w(x,t) = \cos(x + 3t).$$

*12.3.4. Suppose that $u(x,t) = F(x - ct)$. Evaluate:

(a) $\frac{\partial u}{\partial t}(x, 0)$

(b) $\frac{\partial u}{\partial x}(0, t)$

(a) By the chain rule

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} F(x-ct) = -cF'(x-ct)$$

Therefore

$$\frac{\partial u}{\partial t}(x,0) = -cF'(x) = -c\frac{dF(x)}{dx}.$$

(b) Similarly

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} F(x-ct) = F'(x-ct)$$

Therefore

$$\frac{\partial u}{\partial x}(0,t) = F'(-ct) = -\frac{1}{c}\frac{dF(-ct)}{dt},$$