2. Recall the one-dimensional heat equation with constant thermal properties given by

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L].$$

Here c is the heat capacity, ρ the density, K_0 the conductivity, Q the rate of production of heat energy and u the temperature.

(i) Suppose $L = 4$ and $Q/K_0 = x^2(x-4)^2$. If the initial condition and boundary conditions satisfy

$$u(x, 0) = x^2 \quad \text{for} \quad x \in [0, 4]$$
$$u(0, t) = 1 \quad \text{and} \quad u(4, t) = 13 \quad \text{for} \quad t > 0,$$

find the equilibrium temperature of the rod obtained as $t \to \infty$.

The equilibrium solution as $t \to \infty$ doesn't depend on time and consequently $\partial u/\partial t = 0$. This yields the ODE

$$\frac{d^2u}{dx^2} = -x^2(x-4)^2 \quad \text{such that} \quad u(0) = 1 \quad \text{and} \quad u(4) = 13.$$

Integration yields

$$\frac{du}{dx} = \int -x^2(x-4)^2\,dx = \int -x^2(x^2 - 8x + 16)\,dx$$
$$= \int(-x^4 + 8x^3 - 16x^2)\,dx = -\frac{1}{5}x^5 + 2x^4 - \frac{16}{3}x^3 + a$$

and

$$u = \int\left(-\frac{1}{4}x^5 + 2x^4 - \frac{16}{3}x^3 + a\right)dx = -\frac{1}{30}x^6 + \frac{2}{5}x^5 - \frac{4}{3}x^4 + ax + b.$$

Now, solve for a and b using the boundary conditions.

$u(0) = \frac{-1}{30}0^6 + \frac{2}{5}0^5 - \frac{4}{3}0^4 + a0 + b = 1$ implies $b = 1$

$u(4) = \frac{-1}{30}4^6 + \frac{2}{5}4^5 - \frac{4}{3}4^4 + a4 + 1 = 13$ implies

$$a = \frac{1}{4}\left(12 + \frac{1}{30}4^6 - \frac{2}{5}4^5 + \frac{4}{3}4^4\right)$$

$$= 3 + \frac{2}{15}4^4 - \frac{2}{5}4^4 + \frac{1}{3}4^4 = 3 + \left(\frac{2}{15} - \frac{2}{5} + \frac{1}{3}\right)4^4$$

$$= 3 + \frac{1}{15}256 = \frac{45 + 256}{15} = \frac{301}{15}$$

Therefore the equilibrium solution is

$$u(x) = \frac{-1}{30}x^6 + \frac{2}{5}x^5 - \frac{4}{3}x^4 + \frac{301}{15}x + 1.$$

(ii) Suppose $L = 5$ and $Q = 0$. If the initial condition and boundary conditions satisfy

$$u(x,0) = x(5-x) \quad \text{for} \quad x \in [0,5]$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=5} = 0 \quad \text{for} \quad t > 0,$$

find the equilibrium temperature of the rod obtained as $t \to \infty$.

Since $Q=0$ the equilibrium solution satisfies

$$\frac{d^2u}{dt^2} = 0 \quad \text{such that} \quad u'(0)=0 \quad \text{and} \quad u'(5)=0$$

The general solution is $u(x) = ax+b$.

Since $u'(x) = a$ the boundary conditions imply

$$u'(0) = a = 0 \qquad \text{so} \qquad a = 0$$

Since $u'(5)=0$ is automatically satisfied the constant b needs to be solved using something more.

Since $u'(0)=0$ and $u'(5)=0$ are the zero-flux conditions of an insulated boundary, the total heat energy within the bar will be conserved over time. Thus

$$\int_0^5 u(x,0)\,dx = \int_0^5 x(5-x)\,dx = \int_0^5 5x - x^2 = \frac{5}{2}x^2 - \frac{1}{3}x^3 \Big|_0^5$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right)5^3 = \frac{1}{6}5^3$$

must be equal the heat energy in the equilibrium solution

$$\int_0^5 b \, dx = 5b.$$

Thus, $\frac{1}{6} 5^3 = 5b.$

Solving for b yields $b = \frac{25}{6}.$

Therefore the equilibrium solution is

$$u(x) = \frac{25}{6}.$$