From last time

$$c(x)\,\rho(x)\,\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\,\frac{\partial u}{\partial x}\right) + Q(x,t)$$

General heat equation for a 1-D conducting rod.

Heat equation that we study:

$$c(x)\,\rho(x)\,\frac{\partial u(x,t)}{\partial t} = K_0\,\frac{\partial^2 u(x,t)}{\partial x^2}$$

Even simpler assume $c=c(x)$ and $\rho=\rho(x)$ are also const.

$$c\rho\,\frac{\partial u(x,t)}{\partial t} = K_0\,\frac{\partial^2 u(x,t)}{\partial x^2}$$

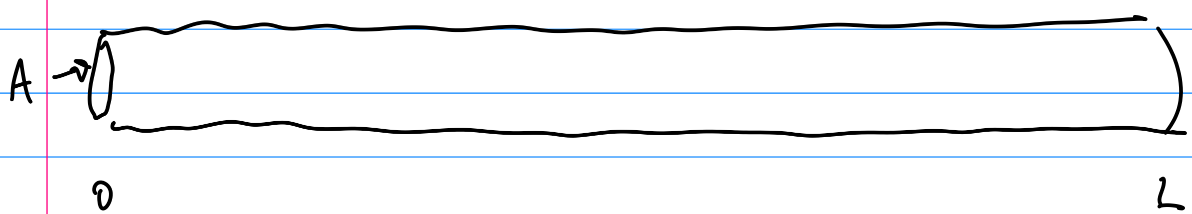
Thus $\boxed{\dfrac{\partial u}{\partial t} = k\,\dfrac{\partial^2 u}{\partial x^2}}$ where $k = \dfrac{K_0}{c\rho}$.

simplest heat equation.

diffusivity

rate of diffusion of the heat.

Conducting rod



- physically need to know what happens at the ends of the domain.. $x=0$ and $x=L$
- what was the initial temperature distribution in the rod,

Physical units in this problem

$$\boxed{\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}}$$

$$\frac{\partial u}{\partial t} \sim \frac{[u]}{[T]} \qquad \frac{\partial^2 u}{\partial x^2} \sim \frac{[u]}{[L]^2}$$

$$\kappa \sim \frac{\cancel{[u]}}{[T]} \Big/ \frac{\cancel{[u]}}{[L]^2} = \frac{[L]^2}{[T]}$$

diffusivity
rate of diffusion of the heat

$$\kappa = \frac{K_0}{c\rho} = \frac{\dfrac{\cancel{[E]}}{[T]\cancel{[L]}\cancel{[u]}}}{\dfrac{\cancel{[E]}}{\cancel{[M]}\cancel{[u]}}\dfrac{\cancel{[M]}}{[L]^{\cancel{3}\,2}}} = \frac{[L]^2}{[T]}$$

dimensional consistency as these are the same...

What are the dimensions of K_0?

$$\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x}$$

$$\phi \sim \frac{[E]}{[T][L]^2}$$

$$\frac{[E]}{[T][L^2]} = [K_0]\frac{[u]}{[L]}$$

$$K_0 \sim \frac{[E]}{[T][L^{\cancel{2}}]} \Big/ \frac{[u]}{\cancel{[L]}} = \frac{[E]}{[T][L][u]}$$

- The PDE $\quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial t^2}$
- The initial conditions.
- The boundary condition

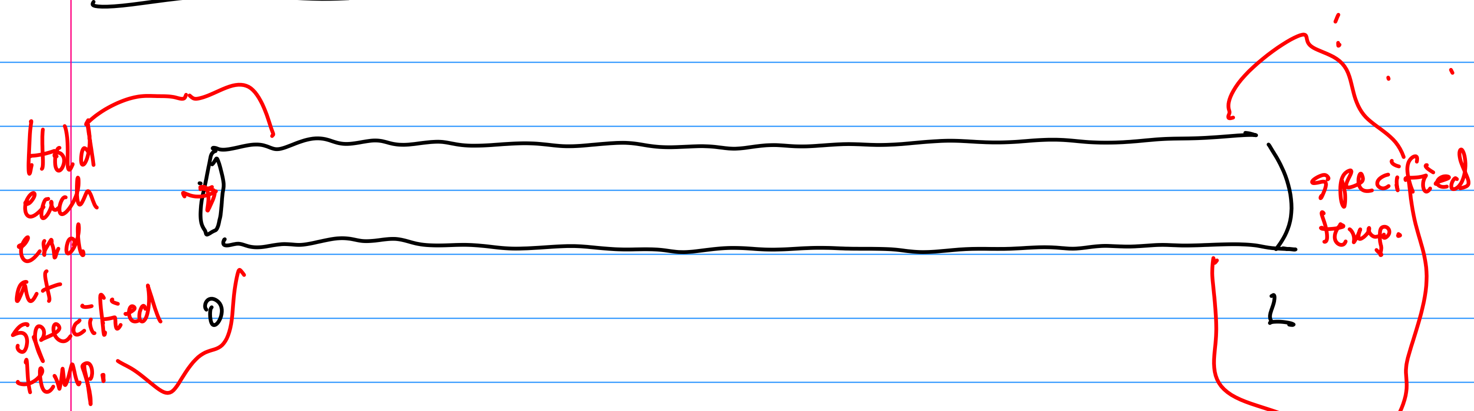
Initial conditions: specify the distribution of heat

for example, $u(x,0) = f(x)$ for $x \in [0,L]$

initial distribution of heat

Boundary Condition:

Heat Bath:



$u(0,t) = g_1(t)$ $\qquad$ $u(L,t) = g_2(t)$

temperature of the heat bath over time...

Insulated boundary condition. No flux at the boundary

$$\phi(0,t) = 0 \qquad \text{and} \qquad \phi(L,t) = 0$$

No flux of heat energy through the boundary.

Since

$$\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x}$$

$$0 = \phi(0,t) = -K_0(0)\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$$

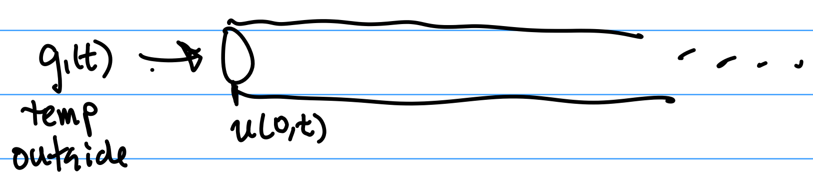
thus $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$

In terms of temperature insulated boundary conditions.

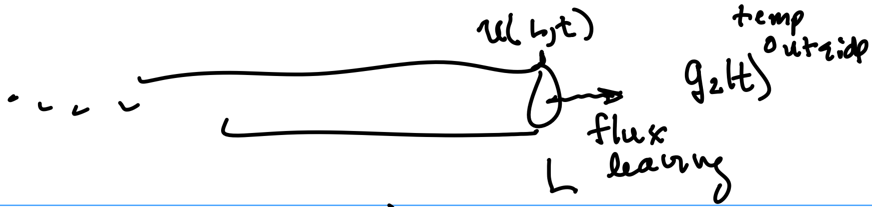
$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \, , \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

Newton's Law of Cooling

and rod temperatures). Experiments show that, as a good approximation, the heat flow leaving the rod is proportional to the temperature difference between the bar and the prescribed external temperature. This boundary condition is called **Newton's law of cooling**. If it is valid at $x = 0$, then



$$-K_0\left.\frac{\partial u}{\partial x}\right|_{x=0} = \phi(0,t) = H_1\left(g_1(t) - u(0,t)\right)$$



$$-K_0 \frac{\partial u}{\partial x}\Big|_{x=L} = \varphi(L,t) = H_2\big(u(L,t) - g_2(t)\big)$$

Goal: predict the future.

Simpler Goal: predict super far into the future as $t \to \infty$.

Idea: as $t \to \infty$ the heat distribution enters into an equilibrium state provided none of the boundary terms have an explicity time dependence...

For example the heat bath...

$u(0,t) = g_1(t)$ — temp at left

$u(L,t) = g_2(t)$ — temp at right

$g_1(t) = g_1$

$g_2(t) = g_2$

constant in time

$u(0,t) = g_1$ $\qquad$ $u(L,t) = g_2$

If in equilibrium then $\frac{\partial u}{\partial t} = 0$ so the heat equation reduces to

$$0 = k \frac{\partial^2 u}{\partial t^2}$$

Solve this next time;