Laplace equation in $[0,W]\times[0,L]$

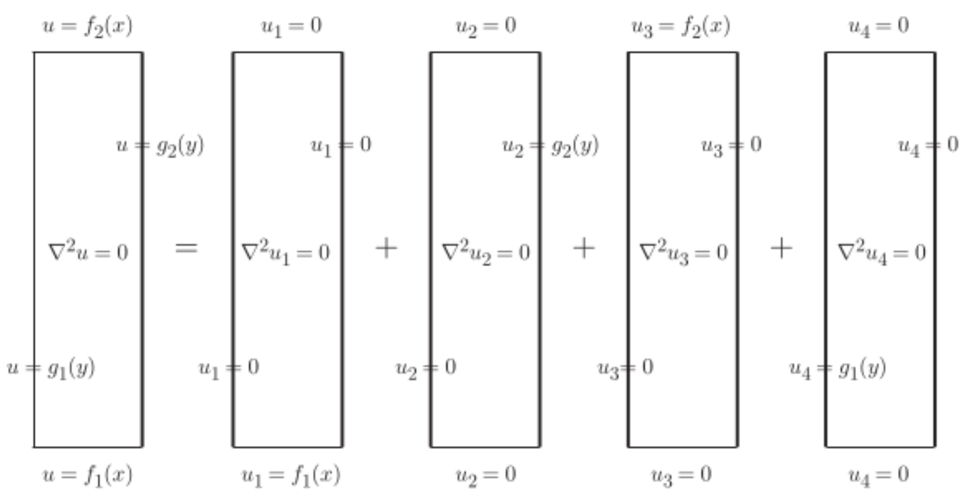
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{for } (x,y)\in[0,W]\times[0,L]$$

with boundary conditions

$$u(x,0) = f_1(x), \quad u(x,H) = f_2(x) \qquad \text{for } x\in[0,W]$$

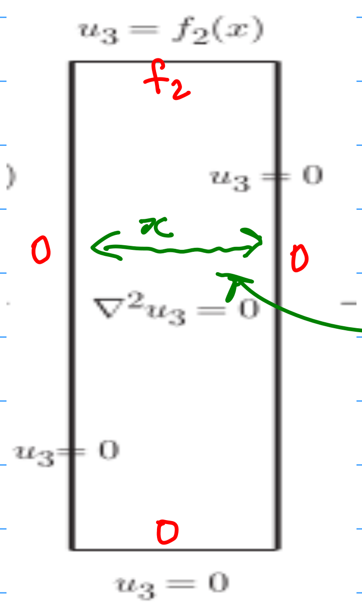
$$u(0,y) = g_1(y), \quad u(W,y) = g_2(y) \qquad \text{for } y\in[0,H]$$

Idea break the boundary value problem down into 4 separate problems



we are on pg 68 right now...

68 Chapter 2 Method of Separation of Variables



in the x direction homogeneous boundary.

Use separation of variables..

$$u(x,y) = \phi(x)\,w(y)$$

identifies which direction has the homogeneous boundary.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x,y) = \varphi(x)\,w(y)$$

Substitute

$$\varphi''(x)\,w(y) + \varphi(x)\,w''(y) = 0$$

Separate variables

$$\frac{w''(y)}{w(y)} = \frac{-\varphi''(x)}{\varphi(x)} = \lambda$$

const, not a function of either x or y.

only a function of y, not a funct of x

only a function of x, not a function of y

Leads to two ODEs

$$w''(y) = \lambda w(y) \qquad \text{and} \qquad \boxed{\begin{aligned} &\varphi''(x) = -\lambda\varphi(x) \\ &\varphi(0) = 0 \\ &\varphi(W) = 0 \end{aligned}}$$

$$w(0) = 0$$

$$w(H) = ? = f_2(x)$$

need to satisfy this boundary using superposition (later).

Solved the ODE for φ already...

$$\varphi''(x) = -\lambda\varphi(x)$$
$$\varphi(0) = 0$$
$$\varphi(W) = 0$$

Know $\lambda > 0$ is the only case when $\varphi \neq 0$ exists...

General solution

$$\varphi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$$\varphi(0) = c_1 = 0 \qquad \text{so} \qquad c_1 = 0$$

$$\varphi(W) = c_2 \sin(\sqrt{\lambda}W) = 0$$

Know $c_2 \neq 0$ otherwise $\varphi = 0$ (which is no good).

$$\sin(\sqrt{\lambda}W) = 0 \qquad \sqrt{\lambda}W = n\pi \qquad n = 1, 2, \ldots$$

Thus $\varphi(x) = c_2 \sin\left(\frac{n\pi}{W}x\right)$

Since
$$\lambda = \frac{n^2\pi^2}{W^2} \qquad n = 1, 2, \ldots$$
so
$$\lambda > 0$$

The other ODE is

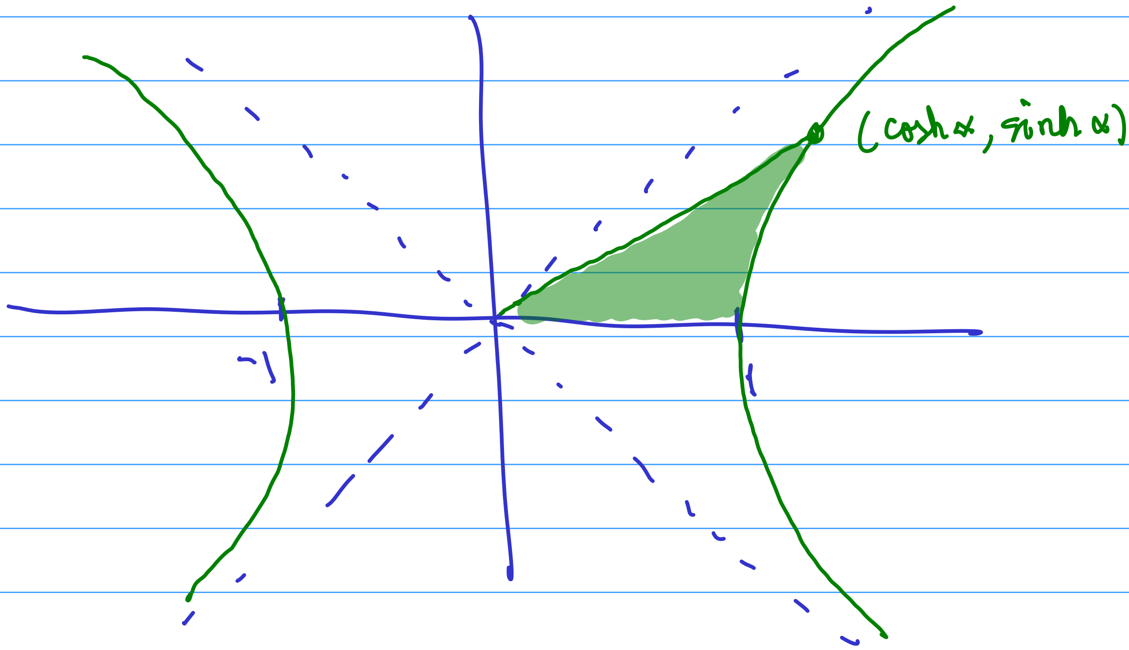
$$w''(y) = \lambda w(y)$$
$$\boxed{w(0) = 0}$$

General solution is

$$w(y) = c_1 e^{\sqrt{\lambda}y} + c_2 e^{-\sqrt{\lambda}y}$$

$$w(y) = c_1 \cosh\sqrt{\lambda}y + c_2 \sinh\sqrt{\lambda}y$$

Note the blue c_1, c_2 are different than the red c_1, c_2



$$w(y) = c_1 e^{\sqrt{\lambda} y} + c_2 e^{-\sqrt{\lambda} y}$$

$$w(y) = c_1 \cosh\sqrt{\lambda} y + c_2 \sinh\sqrt{\lambda} y$$

Recall $\cosh\alpha = \dfrac{e^{\alpha}+e^{-\alpha}}{2}$ $\sinh\alpha = \dfrac{e^{\alpha}-e^{-\alpha}}{2}$

Do the red one now. Try blue at home

$$w(0) = 0 = c_1 \cosh 0 + c_2 \sinh 0 = c_1 = 0$$

$$\boxed{w(y) = c_2 \sinh \frac{n\pi}{W} y}$$