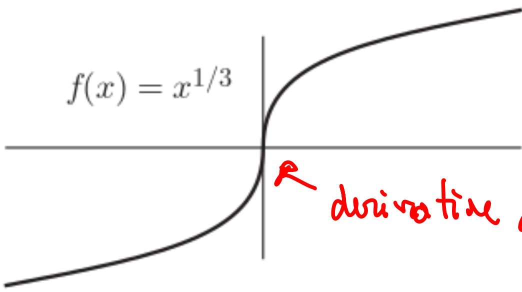
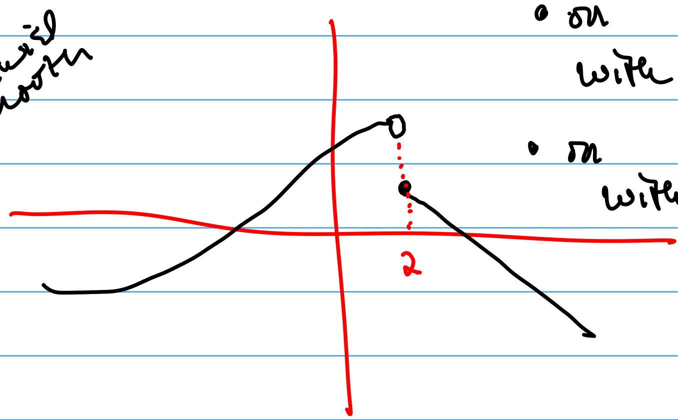
easily, we will discuss only functions $f(x)$ that are piecewise smooth. A function $f(x)$ is **piecewise smooth** (on some interval) if the interval can be broken up into pieces (or sections) such that in each piece the function $f(x)$ is continuous[1] and its derivative df/dx is also continuous. The function $f(x)$ may not be continuous, but the only kind of discontinuity allowed is a finite number of jump discontinuities. A function $f(x)$ has **a jump discontinuity** at a point $x = x_0$ if the limit from the left $\left[f(x_0^-)\right]$ and the limit from the right $\left[f(x_0^+)\right]$ both exist (and are *unequal*), as illustrated in Fig. 3.1.1. An example of a piecewise smooth function is sketched in Fig. 3.1.2. Note that $f(x)$ has two jump

Not piecewise smooth



derivative doesn't exist at the origin

is piecewise smooth



- on $(-\infty, 2)$ is differentiable with continuous derivaton.
- on $[2, \infty)$ is differentiable with continuous derivaton.

$$\text{Fourier series } = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\, dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\, dx. \end{aligned}$$

If $f(x)$ is *piecewise smooth* on the interval $-L \leq x \leq L$, then the Fourier series of $f(x)$ converges

1. to the *periodic extension* of $f(x)$, where the periodic extension is continuous;

What's that?

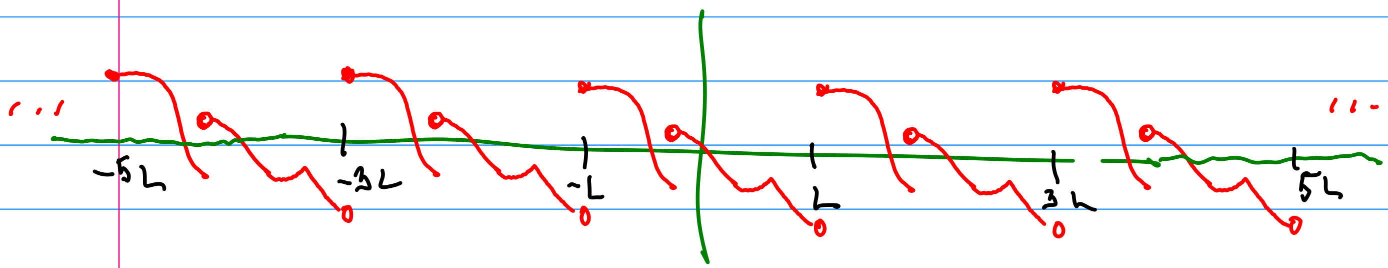
2. to the average of the two limits, usually

$$\frac{1}{2}\left[f(x+) + f(x-)\right],$$

where the periodic extension has a *jump discontinuity.*

What does periodic extension mean?

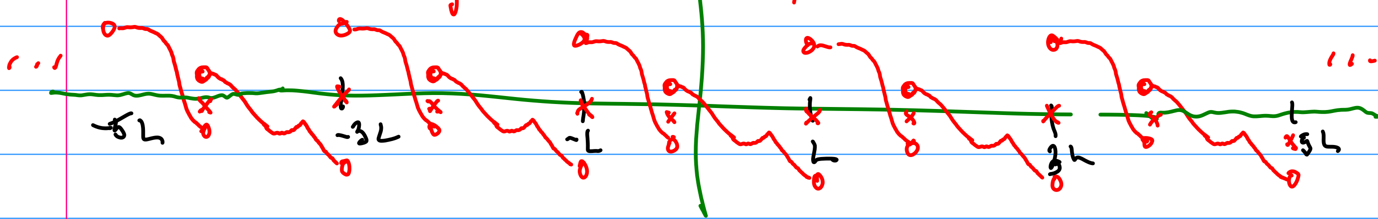
Idea define a function on $[-L, L]$



> 1. Sketch $f(x)$ (preferably for $-L \leq x \leq L$ only).
> 2. Sketch the periodic extension of $f(x)$.

> 3. Mark an "$\times$" at the average of the two values at any jump discontinuity of the periodic extension.

g(x) is the sum of the fourier series



The therem says that

$$\text{Fourier series } = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. = g(x)$$

pointwise convergence...

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\, dx$$

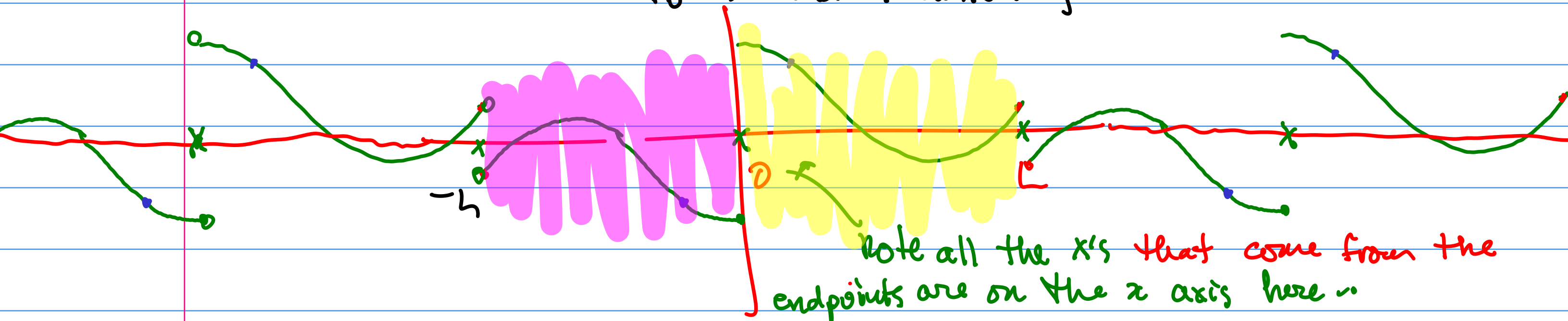
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx.$$

Odd and even extensions: Instead of defining f on $[-L, L]$ define on $[0, L]$ and extend it to $[-L, L]$ so the result is even or odd..

Extend so this fn is odd.

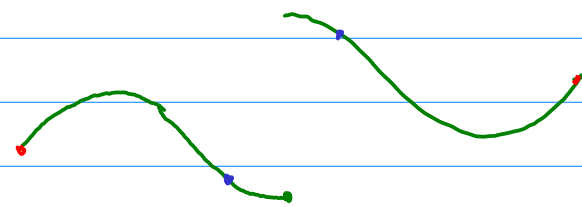
let $f(-x) = -f(x)$ for $x \in [0, L]$.

$f_o(x)$ odd extension of f



(2) extend periodically

(3) put x's at the jumps



$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \approx f_o(x)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_o(x)\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underset{\text{odd}}{f_o(x)} \underset{\text{even}}{\cos\frac{n\pi x}{L}}\,dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\text{odd}}{f_o(x)} \underset{\text{odd}}{\sin\frac{n\pi x}{L}}}_{\text{even function}}\,dx. = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Thus

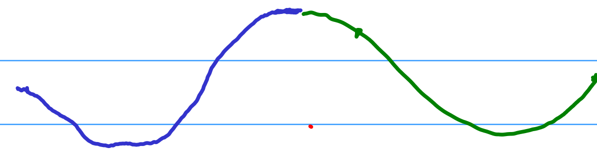
$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Even extension and then extend periodically

in the even case the endpoints automatically match.



Thus,

$$\boxed{a_0+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} \cancel{b_n \sin\frac{n\pi x}{L}}. \approx \boxed{f_e(x)}$$

where

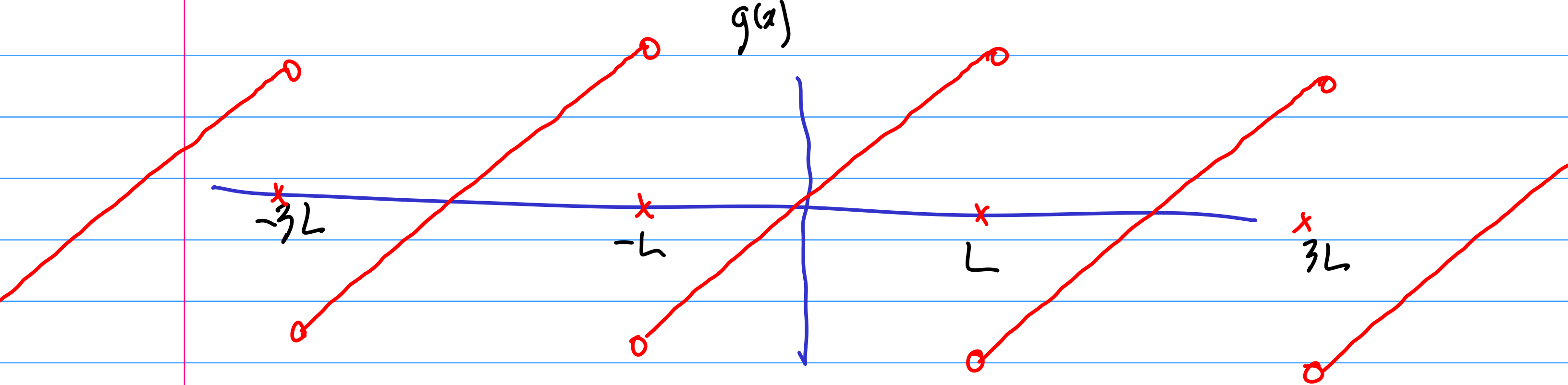
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_e(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \overset{\text{even}}{f_e(x)} \overset{\text{even}}{\cos\frac{n\pi x}{L}}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\overset{\text{even}}{f_e(x)} \overset{\text{odd}}{\sin\frac{n\pi x}{L}}}_{\text{odd function}}\,dx. = 0$$

We want to check whether these Fourier series are solutions to differential equations. We need a theorem about differentiating them...

Example $f(x) = x$ on $[-L, L]$



$$g(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx$$

$u = x$ $\qquad du = dx$

$dv = \sin\frac{n\pi x}{L}$ $\qquad v = -\frac{L}{n\pi}\cos\frac{n\pi x}{L}$

$$= \frac{2}{L}\, x\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)\Big|_0^L - \frac{2}{L}\int_0^L\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}L\left(-\frac{L}{n\pi}\cos n\pi\frac{L}{L}\right) + \frac{2}{L}\int_0^L \frac{L}{n\pi}\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{-2L}{n\pi}\cos n\pi + \frac{2}{n\pi}\,\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L = \frac{-2L}{n\pi}(-1)^n$$

$$g(x) = \sum_{n=1}^{\infty}\left(\frac{-2L}{n\pi}(-1)^n\right)\sin\frac{n\pi x}{L}$$

Derivatives

$$\frac{d}{dx} g(x) = \begin{cases} 1 & \text{for } x \in (-L, L) \\ \text{DNE} & \text{for } x = L \text{ or } -L \end{cases}$$

$$\frac{d}{dx} \sum_{n=1}^{\infty} \left(\frac{-2L}{n\pi} (-1)^n \right) \sin \frac{n\pi x}{L} = \begin{cases} 1 & \text{for } x \in (-L, L) \\ \text{DNE} & \text{for } x = L \text{ or } -L \end{cases}$$

Can I interchange the limits..

Need to do this in order to plug the solutions obtained by superposition into the PDE to check them..

No! since

$$\sum_{n=1}^{\infty} \frac{d}{dx} \left(\frac{-2L}{n\pi} (-1)^n \right) \sin \frac{n\pi x}{L} =$$

$$\sum_{n=1}^{\infty} \left(\frac{-2L}{n\pi} (-1)^n \right) \frac{n\pi}{L} \cos \frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} -2 (-1)^n \cos \frac{n\pi x}{L}$$

The general term doesn't tend to zero so this doesn't converge at all.