Although the Fourier convergence theorem implies pointwise convergence, it doesn't imply uniform convergence...

Example. $f(x) = 100$ on $[0, L]$.

make sine series for this ... so make odd extension


$$g(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \qquad b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

So we think of this as being the Fourier sine series of f and note that by the convergence theorem it converges pointwise to the function g graphed above...

Compute

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx = \frac{2}{L}\int_0^L 100 \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{200}{\cancel{L}}\left(\frac{-\cancel{L}}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_0^L = -\frac{200}{n\pi}\left(\cos n\pi - 1\right)$$

$$= \frac{200}{n\pi}\left(1 - \cos n\pi\right) = \begin{cases} \frac{400}{n\pi} & \text{for } n = 1, 3, 5, \ldots \\ 0 & \text{for } n = 2, 4, 6, \ldots \end{cases}$$

So

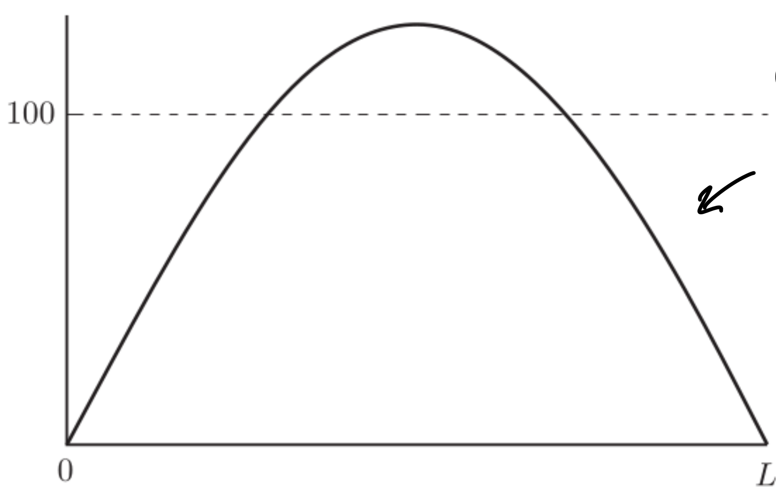
$$b_n = \begin{cases} \dfrac{400}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Therefore

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \sum_{n=1,3,5,\dots} \frac{400}{n\pi} \sin\frac{n\pi x}{L}$$

Again the series converges pointwise for any $x \in \mathbb{R}$ to


Unfortunately it does not converge uniformly...




Graph of

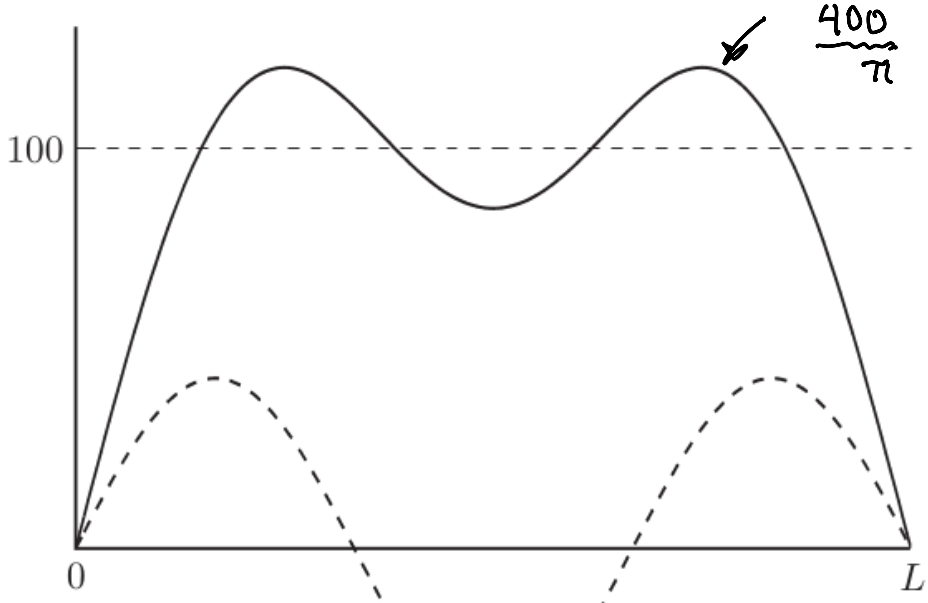
$$\frac{400}{\pi} \sin\frac{\pi x}{L}$$

half a sine wave...

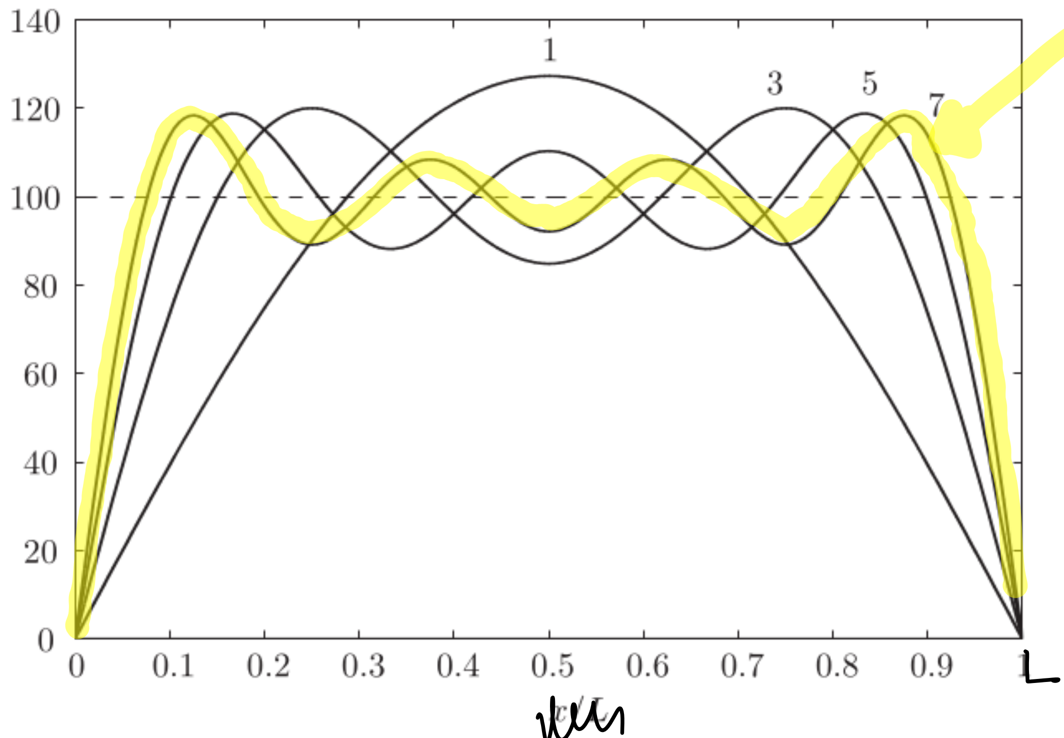
FIGURE 3.3.6 First term of Fourier sine series of $f(x) = 100$.

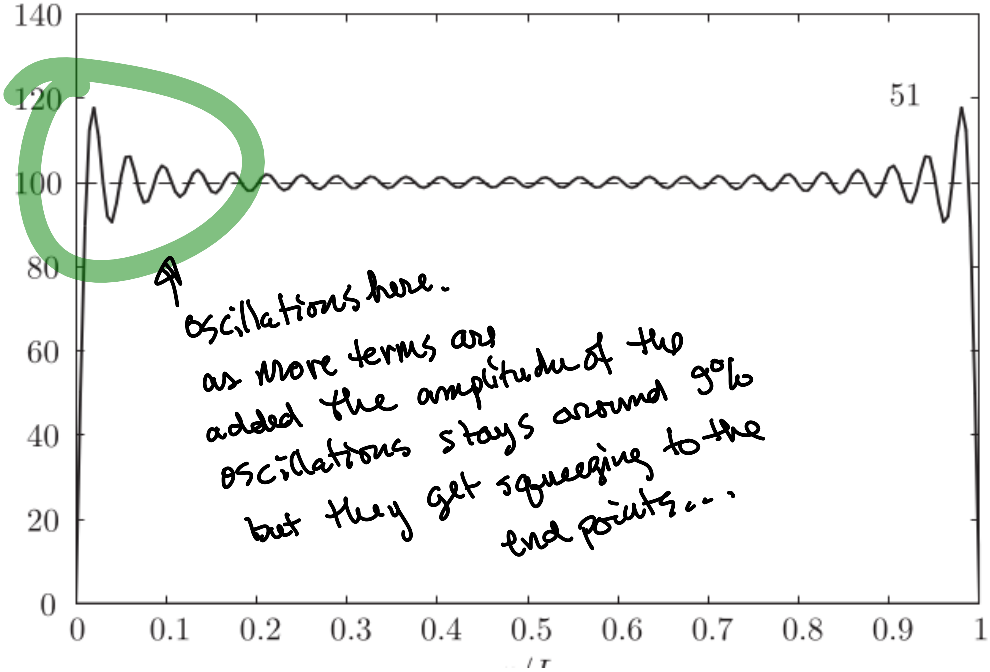
Graph of this

$$\frac{400}{\pi}\sin\frac{\pi x}{L} + \frac{400}{3\pi}\sin\frac{3\pi x}{L}$$




$$\frac{400}{\pi}\sin\frac{\pi x}{L} + \frac{400}{3\pi}\sin\frac{3\pi x}{L} + \frac{400}{5\pi}\sin\frac{5\pi x}{L} + \frac{400}{7\pi}\sin\frac{7\pi x}{L}$$






oscillations here.

as more terms are added the amplitude of the oscillations stays around 9% but they get squeezing to the end points...

51 terms... half are zero so actually 26 sin functions to make this graph...

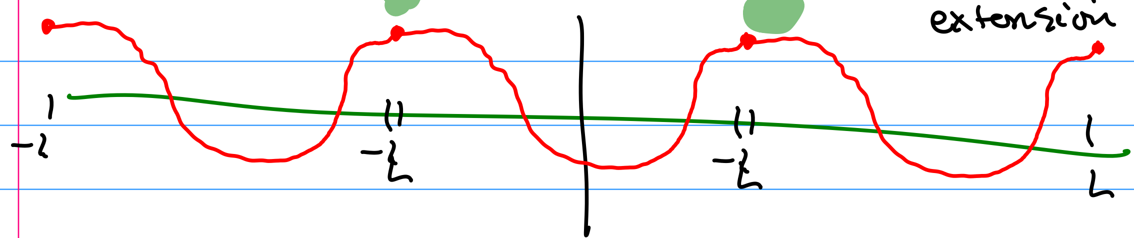
This series does not converge uniformly, so interchanging limiting processes involving $n$ and $x$ are trouble...

> For piecewise smooth $f(x)$, the Fourier series of $f(x)$ is continuous and converges to $f(x)$ for $-L \le x \le L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$.

If you don't add any x's for the jump discontinuities when making g, then

$g(x) = f(x)$ on the domain of f.

means the periodic extension matches up... no jumps so no x's



For piecewise smooth functions $f(x)$, the Fourier sine series of $f(x)$ is continuous and converges to $f(x)$ for $0 < x < L$ if and only if $f(x)$ is continuous and both $f(0) = 0$ *and* $f(L) = 0$.

for an odd extension need endpoints to be zero to avoid jumps and adding x's.

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Differentiation of Fourier series term by term...

1. A Fourier series that is continuous can be differentiated term by term if $f'(x)$ is piecewise smooth.

2. If $f'(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f(-L) = f(L)$.

3. If $f'(x)$ is piecewise smooth, then a continuous Fourier cosine series of $f(x)$ can be differentiated term by term.

4. If $f'(x)$ is piecewise smooth, then the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

5. If $f'(x)$ is piecewise smooth, then a continuous Fourier sine series of $f(x)$ can be differentiated term by term.

6. If $f'(x)$ is piecewise smooth, then the Fourier sine series of a continuous function $f(x)$ can be differentiated term by term only if $f(0) = 0$ and $f(L) = 0$.

Fourier series $g(x)$ $= a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\, dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\, dx. \end{aligned}}$$

$$g'(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \beta_n \sin\frac{n\pi x}{L}$$

$$\alpha_0 = \frac{1}{2L}\int_{-L}^{L} f'(x)\, dx$$

$$\alpha_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\, dx$$

$$\beta_n = \frac{1}{L}\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}\, dx$$

What is the relation between $\alpha$'s $\beta$'s to $a$'s and $b$'s?