Last time we proved

Can we justify term-by-term differentiation with respect to the parameter t? The following theorem states the conditions under which this operation is valid:

The Fourier series of a continuous function $u(x,t)$ (depending on a parameter t)

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} \left[a_n(t) \cos \frac{n\pi x}{L} + b_n(t) \sin \frac{n\pi x}{L} \right]$$

can be differentiated term by term with respect to the parameter t, yielding

$$\frac{\partial}{\partial t} u(x,t) \sim a_0'(t) + \sum_{n=1}^{\infty} \left[a_n'(t) \cos \frac{n\pi x}{L} + b_n'(t) \sin \frac{n\pi x}{L} \right]$$

if $\partial u / \partial t$ is piecewise smooth.

We omit its proof (see Exercise 3.4.7), which depends on the fact that

$$\frac{\partial}{\partial t} \int_{-L}^{L} g(x,t)\, dx = \int_{-L}^{L} \frac{\partial g}{\partial t} dx$$

by working exercise 3.4.7 in the book.

HW4 due Friday, Mar 13
Turn in 3.3.2c
Practice 3.2.1cdg, 3.2.2be, 3.3.1cb, 3.3.7, 3.4.1ab, 3.5.1abc

Read section 3.5 in preparation for Friday...

Proof on integrating Fourier series. Consider

Application of the theory on Fourier series to PDEs.

Heat equation: $u_t = k\, u_{xx}$ for $t \geq 0$ and $x \in [0,L]$

B.C.: $u_x(0,t) = 0$ $u_x(L,t) = 0$ (insulated boundary)

I.C.: $u(x,0) = f(x)$

Recall separation of variables $u(x,t) = \varphi(x) G(t)$

$$\varphi(x) G'(t) = k\varphi''(x) G(t)$$

$$\frac{G'(t)}{kG(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

(constant since left side depends on t only and right depends on x)

Then

$$G'(t) = -\lambda k G(t) \qquad \text{and} \qquad \varphi''(x) = -\lambda \varphi(x)$$

solve this ODE

$$\varphi'(0) = 0 \qquad \varphi'(L) = 0$$

general solution $\varphi(x) = a\cos\sqrt{\lambda}\, x + b\sin\sqrt{\lambda}\, x$

$$\varphi'(x) = -a\sqrt{\lambda}\sin\sqrt{\lambda}\, x + b\sqrt{\lambda}\cos\sqrt{\lambda}\, x$$

$$\varphi'(0) = b\sqrt{\lambda} = 0 \quad \text{so} \quad b = 0$$

$$\varphi'(L) = -a\sqrt{\lambda}\sin\sqrt{\lambda}\, L = 0$$

so $\sqrt{\lambda}\, L = \pi n$ for $n = 1, 2, \ldots$

Therefore define $\sqrt{\lambda} = \frac{n\pi}{L}$

$$\varphi_n(x) = a_n \cos\frac{n\pi x}{L} \qquad \varphi_0(x) = a_0$$

Then solve the ODE for the $G_n(t)$ and write the superposition

$$u(x,t) = \sum_{n=0}^{\infty} \varphi_n(t)\, G_n(t)$$

← the solution..

Justify this using the theory... Idea plug in the answer use the term-by-term differentiation theorems and show that it works.

Given $u_t = k\, u_{xx}$ consider a series solution

plug in

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos\frac{n\pi x}{L}$$

← Fourier series or eigenfunction expansion of the φ ODE.

need to justify term by term differentiation of the series.

$u_t = k u_{xx}$

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{n\pi x}{L}$$

use these two

A Fourier series that is continuous can be differentiated term by term if $f'(x)$ is piecewise smooth.

If $f'(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f(-L) = f(L)$.

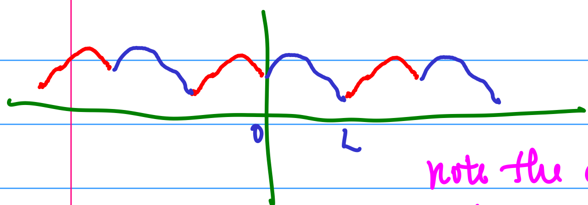
If $f'(x)$ is piecewise smooth, then a continuous Fourier cosine series of $f(x)$ can be differentiated term by term.

If $f'(x)$ is piecewise smooth, then the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

If $f'(x)$ is piecewise smooth, then a continuous Fourier sine series of $f(x)$ can be differentiated term by term.

If $f'(x)$ is piecewise smooth, then the Fourier sine series of a continuous function $f(x)$ can be differentiated term by term only if $f(0) = 0$ and $f(L) = 0$.

recall cosine series can come from an even extension of f



note the even extension always matches up at the pieces....

Need u, u_x to be continuous and u_{xx} to be piecewise smooth. in order to plug the series in to $\frac{\partial^2}{\partial x^2}$ and differentiate term by term

① Assume u is cont. and u_x piecewise smooth the we can differentiate term by term

$$\frac{\partial}{\partial x} u(x,t) = \sum_{n=0}^{\infty} a_n(t) \frac{\partial}{\partial x} \cos \frac{n\pi x}{L} = \sum_{n=1}^{\infty} -\frac{n\pi}{L} a_n(t) \sin \frac{n\pi x}{L}$$

② Assume u_x is cont. and u_{xx} is piecewise smooth

sin series

$f(0) = 0$ and $f(L) = 0$. → means $u_x(0,t) = 0$ and $u_x(L,t) = 0$ } These are the boundary conditions

so the solution $u(x,t)$ automatically satisfies these...

To differentiate term by term in t need u_t to be piecewise smooth...

$$u_t = k u_{xx}$$

$$\underbrace{\sum_{n=0}^{\infty} a_n'(t) \cos\frac{n\pi x}{L} = \sum_{n=1}^{\infty} -k\frac{n^2\pi^2}{L^2} a_n(t) \cos\frac{n\pi x}{L}}_{\text{since cosine orthogonal}}$$

Thus
$$a_0'(t) = 0$$
$$a_n'(t) = -k\frac{n^2\pi^2}{L^2} a_n(t) \qquad \text{for } n = 1, 2, \ldots$$

& (This is the same ODE as for G last time)

Note by plugging in the series expansion into the PDE we can handle heat sources.

$$u_t = k u_{xx} + q(x,t)$$
← heat source..

$$\sum_{n=0}^{\infty} a_n'(t) \cos\frac{n\pi x}{L} = \sum_{n=1}^{\infty} -k\frac{n^2\pi^2}{L^2} a_n(t) \cos\frac{n\pi x}{L} + q(x,t)$$

Next thing is to write $q(x,t)$ using the same eigenfunctions

$$q(x,t) = \sum_{n=0}^{\infty} q_n(t) \cos\frac{n\pi x}{L}$$

$$\sum_{n=0}^{\infty} a_n'(t) \cos\frac{n\pi x}{L} = \sum_{n=1}^{\infty} -k\frac{n^2\pi^2}{L^2} a_n(t) \cos\frac{n\pi x}{L} + \sum_{n=0}^{\infty} q_n(t) \cos\frac{n\pi x}{L}$$

ODE: $a_n'(t) = -\frac{k n^2\pi^2}{L} a_n(t) + q_n(t)$ (solution w/ heat source)