A Fourier series of piecewise smooth $f(x)$ can always be integrated term by term, and the result is a *convergent* infinite series that always *converges* to the integral of $f(x)$ for $-L \le x \le L$ (even if the original Fourier series has jump discontinuities).

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

So if f is piecewise smooth we know the Fourier series converges pointwise to the periodic extension of f with the jump discontinuities replaced by the average value across each jump.

$$G(x) = \int_{-L}^{x} g(s)\,ds = \int_{-L}^{x}\left(a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi s}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi s}{L}\right)ds$$

$$= a_0(x+L) + \sum_{n=1}^{\infty} a_n \int_{-L}^{x} \cos\frac{n\pi s}{L}\,ds + \sum_{n=1}^{\infty} b_n \int_{-L}^{x} \sin\frac{n\pi s}{L}\,ds$$

$$\int_{-L}^{x} \cos\frac{n\pi s}{L}\,ds = \frac{L}{n\pi}\sin\frac{n\pi s}{L}\Big|_{-L}^{x} = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$$

$$\int_{-L}^{x} \sin\frac{n\pi s}{L}\,ds = -\frac{L}{n\pi}\cos\frac{n\pi s}{L}\Big|_{-L}^{x} = \frac{-L}{n\pi}\left(\cos\frac{n\pi x}{L} - \cos n\pi\right)$$

$$= \frac{-L}{n\pi}\cos\frac{n\pi x}{L} + \frac{L}{n\pi}(-1)^n$$

$$G(x) = a_0(x+L) + \sum_{n=1}^{\infty} b_n \frac{L}{n\pi}(-1)^n + \sum_{n=1}^{\infty} b_n \frac{-L}{n\pi} \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} a_n \frac{L}{n\pi} \sin\frac{n\pi x}{L}$$

$$G(x) = \int_{-L}^{x} g(s)\,ds.$$

Converges because its alternating with general term going to zero...

Compare with the original Fourier series...

note that since there is another n in the denominator these converge even better than before

recall

$$\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

CHAPTER 4

Wave Equation: Vibrating Strings and Membranes

What have we done so far?

	Type
① Heat equation... $u_t = k\, u_{xx}$	parabolic equation. (time evolution of something with dissipation) dynamics problem
② Laplace equation $u_{xx} + u_{yy} = 0$	elliptic equation. (stationary state of the 2D heat equation) statics problem
③ Wave equation. $$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},}$$	hyperbolic equation (time evolution of a conservation law) dynamics problem

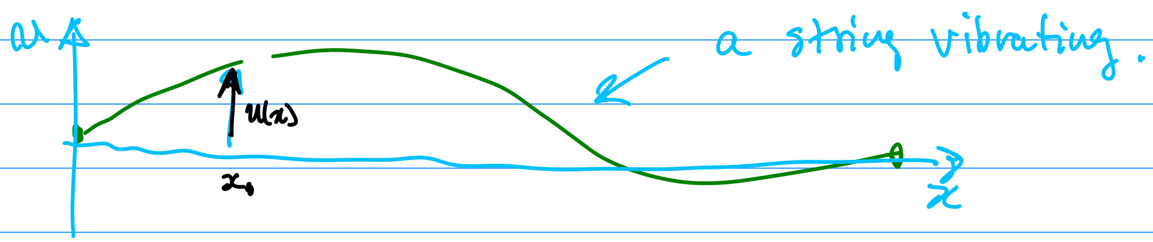
Derivation of the wave equation... by Newton's law

$$F = ma$$

force

mass

acceleration.

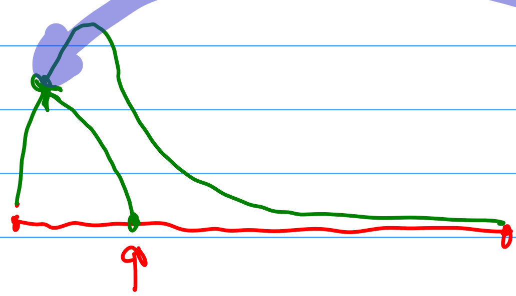


Let $u(x,t)$ be the dispacement of the string from its equilibrium position at time t.

velocity is $\quad v(x,t) = \dfrac{\partial u(x,t)}{\partial t}$

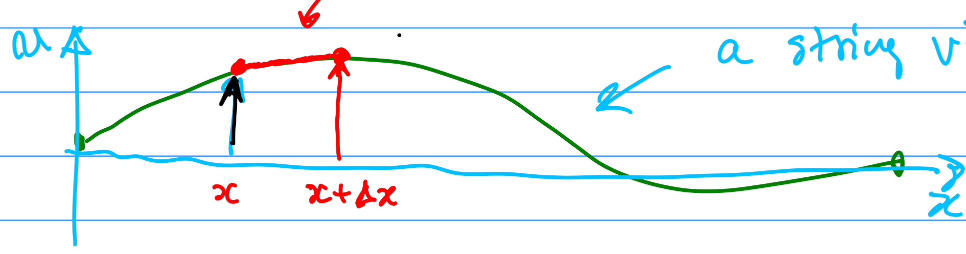
acceleration is $\quad a(x,t) = \dfrac{\partial v(x,t)}{\partial t}$

Assume oscillations are small so the only displacement are in the vertical direction..



the point on the string here moved to here when there was a large displacement of the string... Assume this doesnt happen..

Small displacem en



Let ρ be the density of the string.

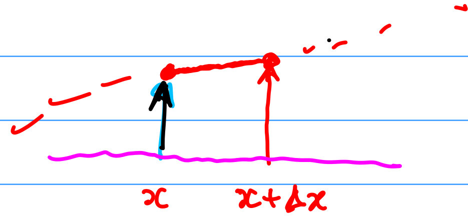
$$m = \int_{x}^{x+\Delta x} \rho(x)\,dx$$

Suppose density is constant. Then

$$m = \rho \Delta x$$

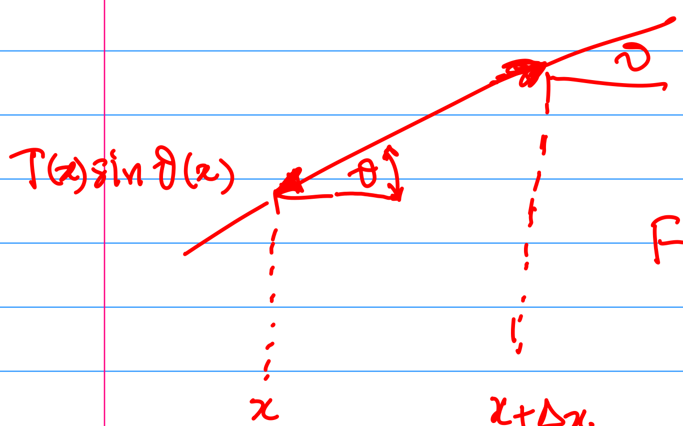
Therefore $F = ma = \rho \Delta x \dfrac{\partial^2 u}{\partial t^2}$

What about the force?



Let T be the tension in the string

Since we assume the motion is up and down we neglect the left and right components of the tension (force).



$$F = T(x+\Delta)\sin\theta(x+\Delta) - T(x)\sin\theta(x)$$