2.5#2.

2.5.2. Consider $u(x,y)$ satisfying Laplace's equation inside a rectangle ($0 < x < L$, $0 < y < H$) subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial y}(x,0) = 0$$

$$\frac{\partial u}{\partial x}(L,y) = 0, \qquad \frac{\partial u}{\partial y}(x,H) = f(x).$$

*(a) *Without* solving this problem, briefly explain the physical condition under which there is a solution to this problem.

(b) Solve this problem by the method of separation of variables. Show that the method works only under the condition of part (a). [*Hint*: You may use (2.5.16) without derivation.]

(c) The solution [part (b)] has an arbitrary constant. Determine it by consideration of the time-dependent heat equation (1.5.11) subject to the initial condition

$$u(x,y,0) = g(x,y).$$

(a) Since u solves Laplace equation thus

$$u_{xx} + u_{yy} = 0$$

Now integrate the equation over the whole domain $x \in [0,L]$ and $y \in [0,H]$

$$\int_0^H \int_0^L (u_{xx} + u_{yy})\, dx\, dy = \int_0^H \int_0^L 0\, dx\, dy = 0$$

$$\|$$

$$\int_0^H \left(\int_0^L u_{xx}\, dx \right) dy + \int_0^L \left(\int_0^H u_{yy}\, dy \right) dx$$

divergence theorem for a square

$$= \int_0^H \left(\underbrace{u_x \Big|_{x=0}^{L}}_{\text{value on boundary}} \right) dy + \int_0^L \left(u_y \Big|_{y=0}^{H} \right) dx$$

values on boundary

recall

$$\frac{\partial u}{\partial x}(0,y)=0, \quad \frac{\partial u}{\partial y}(x,0)=0$$
$$\frac{\partial u}{\partial x}(L,y)=0, \quad \frac{\partial u}{\partial y}(x,H)=f(x).$$

$$0 = \int_0^H \left(u_x \Big|_{x=0}^{L} \right) dy + \int_0^L \left(u_y \Big|_{0}^{H} \right) dx$$

$$= \int_0^H (0-0)\, dy + \int_0^L (f(x)-0)\, dx = \int_0^L f(x)\, dx$$

(b) Solve using separation of variables

PDE $\quad u_{xx} + u_{yy} = 0$

B.C.

$$\frac{\partial u}{\partial x}(0,y)=0, \quad \frac{\partial u}{\partial y}(x,0)=0$$
$$\frac{\partial u}{\partial x}(L,y)=0, \quad \frac{\partial u}{\partial y}(x,H)=f(x).$$

homogeneous $\qquad$ not homogeneous

Let $\quad u(x,y) = \varphi(x)\, h(y) \quad$ and plug in

$$\varphi''(x)\, h(y) + \varphi(x)\, h''(y) = 0$$

Separate x from y to obtain

$$-\frac{h''(y)}{h(y)} = \boxed{\frac{\varphi''(x)}{\varphi(x)} = -\lambda}$$

for convenience use $-\lambda$ here so that $\lambda \geq 0$ later.

The ODEs

$$h''(y) = \lambda h(y) \qquad \text{and} \qquad \varphi''(x) = -\lambda \varphi(x)$$

$h'(0)=0 \qquad h'(H) = ?$ $\qquad \varphi'(0)=0 \qquad \varphi'(L)=0$

by superposition

Solve $\varphi''(x) = -\lambda \varphi(x)$

$\varphi'(0)=0 \quad \varphi'(L)=0$

Case $\lambda=0$: $\varphi''(x) = 0$

General solution $\varphi(x) = c_1 x + c_2$

$\varphi'(x) = c_1 \quad$ so $\quad c_1 = 0$

eigen function: $\varphi(x) = c_2$

Case $\lambda > 0$: $\varphi'' = -\lambda \varphi(x)$

General solution: $\varphi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$\varphi'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$

$\varphi'(0) = c_2\sqrt{\lambda} = 0$ implies $c_2 = 0$

$\varphi'(L) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, L = 0$

Since $c_1 = 0$ implies $\varphi \equiv 0$ then $\sin\sqrt{\lambda}\, L = 0$ in which case $\sqrt{\lambda}\, L = n\pi$ for $n = 1, 2, \dots$

Thus, $\sqrt{\lambda} = \frac{n\pi}{L} \qquad \lambda = \frac{n^2\pi^2}{L^2}$

eigenfunction: $\varphi(x) = c_1 \cos\frac{n\pi x}{L}$

Case $\lambda < 0$: No non-zero solutions that satisfy the boundary condition.

Now solve $h''(y) = \lambda h(y)$

$h'(0) = 0 \qquad h'(H) = ?$

Thus $\quad h''(y) = \frac{n^2\pi^2}{L^2} h(y) \quad$ for $\quad n = 0, 1, 2, \ldots$

Case $n = 0$:

General solution $\quad h(y) = c_1 y + c_2$

$$h'(y) = c_1$$

$$h'(0) = c_1 = 0 \quad \text{so} \quad h(y) = c_2$$

Case $n = 1, 2, \ldots$

General solution $\quad h(y) = c_1 \cosh \frac{n\pi y}{L} + c_2 \sinh \frac{n\pi y}{L}$

$$h'(y) = c_1 \frac{n\pi}{L} \sinh \frac{n\pi y}{L} + c_2 \frac{n\pi}{L} \cosh \frac{n\pi y}{L}$$

$$h'(0) = c_2 \frac{n\pi}{L} = 0 \qquad c_2 = 0$$

So $\quad h(y) = c_1 \cosh \frac{n\pi y}{L}$

The superposition is

$$u(x,y) = b_0 + \sum_{n=1}^{\infty} b_n \cos \frac{n\pi x}{L} \cosh \frac{n\pi y}{L}$$

Now satisfy the final boundary condition. $\frac{\partial u}{\partial y}(x, H) = f(x).$

$$u_y(x,H) = \boxed{0} + \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos \frac{n\pi x}{L} \sinh \frac{n\pi H}{L} = f(x)$$

Solve for b_n using orthogonality. Thus,

$$\frac{\cancel{L}}{2} \cdot b_n \frac{n\pi}{\cancel{L}} \sinh \frac{n\pi H}{L} = \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{2}{n\pi \sinh \frac{n\pi H}{L}} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

Note that if I integrate this over $[0,L]$ then

$$\boxed{0} + \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos \frac{n\pi x}{L} \sinh \frac{n\pi H}{L} = f(x)$$

$$\int_0^L 0\, dx + \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \left(\int_0^L \cos \frac{n\pi x}{L}\, dx \right) \sinh \frac{n\pi H}{L} = \int_0^L f(x)\, dx$$

$$0 + \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \left(0 \right) \sinh \frac{n\pi H}{L} = \int_0^L f(x)\, dx$$

So we recover the condition $\int_0^L f(x)\, dx = 0$ in order to solve for the coefficients since the constant term is missing from the cosine series.

(c) How to determine the b_0?

(c) The solution [part (b)] has an arbitrary constant. Determine it by consideration of the time-dependent heat equation (1.5.11) subject to the initial condition

$$u(x, y, 0) = g(x, y).$$

initial amount of heat

$$\int_0^L \int_0^H g(x,y)\, dy\, dx = \int_0^L \int_0^H \left(b_0 + \cancel{\sum_{n=1}^{\infty} b_n \cos \frac{n\pi x}{L} \cosh \frac{n\pi y}{L}} \right) dy\, dx$$

0

$$b_0 = \frac{1}{L} \frac{1}{H} \int_0^L \int_0^H g(x,y)\, dy\, dx.$$

Similar to 14#7

3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L]$$

subject to the homogeneous boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

Solve the initial value problem if the temperature is initially

$$u(x,0) = 5\sin\left(\frac{3\pi x}{2L}\right).$$

Separation of variables $u(x,t) = \varphi(x)G(t)$ substitute

$$\varphi(x)G'(t) = k\varphi''(x)G(t)$$

Thus

$$\frac{G'(t)}{kG(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

gives the ODEs

$$G'(t) = k\lambda G(t) \quad \text{and} \quad \varphi''(x) = -\lambda\varphi(x)$$

$$G(0) = ? \qquad \varphi(0) = 0 \qquad \varphi'(L) = 0$$

by superposition

Find eigenfunctions:

Case $\lambda = 0$ $\quad \varphi''(x) = 0$ so $\varphi(x) = c_1 x + c_2$

General solution

$$\varphi(0) = c_2 = 0 \qquad \varphi'(x) = c_1 \qquad \varphi'(L) = c_1 = 0$$

No eigenfunction when $\lambda = 0$

Case $\lambda > 0$ general solution

$$\varphi(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$$

$\varphi(0) = c_1 = 0$ so $c_1 = 0$

$\varphi'(x) = c_2 \sqrt{\lambda} \cos \sqrt{\lambda} x$

$\varphi'(L) = c_2 \sqrt{\lambda} \cos \sqrt{\lambda} L = 0$

Thus $\cos \sqrt{\lambda} L = 0$ or $\sqrt{\lambda} L = \frac{\pi}{2} + n\pi$ for $n = 0, 1, 2, \ldots$

$$\sqrt{\lambda} = \frac{\frac{\pi}{2} + n\pi}{L} = \frac{(n+\frac{1}{2})\pi}{L}$$

$$\lambda = \frac{(n+\frac{1}{2})^2 \pi^2}{L^2}$$

Solve the

$$h'(t) = -k\lambda h(t).$$

beyond here was finished after class ...

General solution $h(t) = c_1 e^{-k\lambda t}$

By superposition write

$$u(x,t) = \sum_{n=0}^{\infty} a_n \sin \frac{(n+1/2)\pi x}{L} e^{-k\frac{(n+1/2)^2\pi^2}{L^2} t}$$

Then solve for the a_n such that

$$u(x,0) = \sum_{n=0}^{\infty} a_n \sin \frac{(n+1/2)\pi x}{L} = 5 \sin \frac{3\pi x}{2L}.$$

Since $n = 1$ corresponds to $\sin \frac{3\pi x}{2L}$ then we see that $a_1 = 5$ and the other coefficients in the series are all zero. It follows that

$$u(x,t) = 5 \sin \frac{3\pi x}{2L} e^{-k \cdot 9\pi^2 t/(4L^2)}$$