Let $A \in \mathbb{R}^{n \times n}$ and $x, y \in \mathbb{R}^n$

$$x \cdot Ay = \sum_{i=1}^{n} x_i [Ay]_i$$

$$[Ay]_i = \sum_{j=1}^{n} A_{ij} y_j$$

Thus

$$x \cdot Ay = \boxed{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i A_{ij} y_j}$$

$$A^T x \cdot y = \sum_{p=1}^{n} [A^T x]_p y_p$$

$$[A^T x]_p = \sum_{q=1}^{n} [A^T]_{pq} x_q = \sum_{q=1}^{n} A_{qp} x_q$$

Thus

$$A^T x \cdot y = \sum_{p=1}^{n} \sum_{q=1}^{n} A_{qp} x_q y_p = \boxed{\sum_{q=1}^{n} \sum_{p=1}^{n} x_q A_{qp} y_p}$$

The reason A^T is useful is because $x \cdot Ay = A^T x \cdot y$

this is where A^T comes from.

Back to PDE...

PDE: $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} = Lu \qquad \text{where } L = k\frac{\partial^2}{\partial x^2}$$

or L^*

BC: $u(0,t)=0$
$u(L,t)=0$

Need a dot product:

$$f \cdot g = (f,g) = \int_0^L f(x)\,g(x)\,dx$$

If $(f, Lg) = (Lf, g)$ then $L^\dagger = L$

assume $f(0)=0$, $f(L)=0$, $g(0)=0$, $g(L)=0$

$$(f,Lg) = \int_0^L f(x)\,Lg(x)\,dx = \int_0^L f(x)\; k\frac{d^2}{dx^2}g(x)\,dx$$

$$= k\int_0^L f(x)\,g''(x)\,dx$$

are these the same?

$$(Lf, g) = k\int_0^L f''(x)\,g(x)\,dx$$

By integration by parts

$$\int_0^L f(x)\,g''(x)\,dx = f(x)g'(x)\Big|_0^L - \int_0^L g'(x)\,f'(x)\,dx$$

$u = f(x)$ $\qquad du = f'(x)\,dx$

$dv = g''(x)\,dx$ $\qquad v = g'(x)$

Therefore

$$\int_0^L f(x)\, g''(x)\, dx = \boxed{-\int_0^L g'(x)\, f'(x)\, dx}$$

Also

$$\int_0^L f''(x) g(x)\, dx = \cancel{g(x) f'(x) \Big|_0^L} - \int_0^L f'(x) g'(x)\, dx$$

$u = g(x)$ $\qquad du = g'(x)\, dx$

$dv = f''(x)\, dx$ $\qquad v = f'(x)$

Same

Therefore

$$\int_0^L f''(x) g(x)\, dx = \boxed{-\int_0^L f'(x) g'(x)\, dx}$$

By definition $L^\dagger$ is the operator (differential operator) such that

$(f, Lg) = (L^\dagger f, g)$ for all f, g that satisfy the B.C.s.

We showed that

$(f, Lg) = (Lf, g)$ using integration by parts.

Conclusion: $L = L^\dagger$ i.e. that $\boxed{L \text{ is self adjoint}}$

Theorem: Suppose

ODE that comes from PDE using separation of variables

eigenvalues

PDE

$$\boxed{\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi}_{} + \lambda\sigma(x)\phi = 0,}$$

$a < x < b,$

$L = \frac{d}{dx} p(x) \frac{d}{dx} + q(x)$

$L\phi = \underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi}_{}$

BC

$$\boxed{\begin{aligned} \beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) &= 0 \\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) &= 0, \end{aligned}}$$

and $p > 0$ and $\sigma > 0$

1. All the eigenvalues λ are real.
2. There exist an infinite number of eigenvalues:
$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots .$$
   a. There is a smallest eigenvalue, usually denoted λ_1.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$
Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$
6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:
$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2\sigma\,dx},$$
where the boundary conditions may somewhat simplify this expression.

$$L = \frac{d}{dx} p(x) \frac{d}{dx} + q(x)$$

Claim L is self adjoint. Thus $(f, Lg) = (Lf, g)$ for all f and g satisfying the boundary conditions...

$$\beta_1 f(a) + \beta_2 \frac{df}{dx}(a) = 0$$
$$\beta_3 f(b) + \beta_4 \frac{df}{dx}(b) = 0,$$

$$\beta_1 g(a) + \beta_2 \frac{dg}{dx}(a) = 0$$
$$\beta_3 g(b) + \beta_4 \frac{dg}{dx}(b) = 0,$$

$$(f, Lg) = \int_0^L f(x)\, Lg(x)\, dx = \int_0^L f(x) \left[\frac{d}{dx}\left(p(x) \frac{dg(x)}{dx} \right) + q(x)\, g(x) \right] dx$$

compared with

okay

$$(Lf, g) = \int_0^L Lf(x)\, g(x)\, dx = \int_0^L \left[\frac{d}{dx}\left(p(x) \frac{df(x)}{dx} \right) + q(x)\, f(x) \right] g(x)\, dx.$$

Integration by parts

$$\int_0^L f(x) \frac{d}{dx}\left(p(x) \frac{dg(x)}{dx} \right) dx = f(x)\, p(x)\, g'(x) \Big|_0^L - \boxed{\int_0^L p(x) \frac{dg(x)}{dx} f'(x)\, dx}$$

$u = f(x)$ $\qquad$ $du = f'(x)\, dx$

$dv = \frac{d}{dx}\left(p(x) \frac{dg(x)}{dx} \right) dx$ $\qquad$ $v = p(x) \frac{dg(x)}{dx}$

also

same

$$\int_0^L \frac{d}{dx}\left(p(x) \frac{df(x)}{dx} \right) g(x)\, dx = g(x)\, p(x)\, f'(x) \Big|_0^L - \boxed{\int_0^L p(x)\, f'(x)\, g'(x)\, dx}$$

$u = g(x)$ $\qquad$ $du = g'(x)$

$dv = \frac{d}{dx}\left(p(x) \frac{df(x)}{dx} \right) dx$ $\qquad$ $v = p(x) \frac{df(x)}{dx}$

What's left are the boundary terms: Are these equal?

$$f(x)\,p(x)\,g'(x)\Big|_0^L \stackrel{?}{=} g(x)\,p(x)\,f'(x)\Big|_0^L$$

Boundary conditions

$$\beta_2 f'(0) = -\beta_1 f(0) \qquad \beta_2 g'(0) = -\beta_1 g(0)$$

$$\beta_4 f'(L) = -\beta_3 f(L) \qquad \beta_4 g'(L) = -\beta_3 g(L)$$

$$f(x)\,p(x)\,g'(x)\Big|_0^L = f(L)\,p(L)\left(-\frac{\beta_3}{\beta_4}g(L)\right) - f(0)\,p(0)\left(-\frac{\beta_1}{\beta_2}g(0)\right)$$

same   same

$$g(x)\,p(x)\,f'(x)\Big|_0^L = g(L)\,p(L)\left(-\frac{\beta_3}{\beta_4}f(L)\right) - g(0)\,p(0)\left(-\frac{\beta_1}{\beta_2}f(0)\right)$$