$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)+q(x)\phi+\lambda\sigma(x)\phi=0, \qquad a<x<b,$$

$$\beta_1\phi(a)+\beta_2\frac{d\phi}{dx}(a)=0$$
$$\beta_3\phi(b)+\beta_4\frac{d\phi}{dx}(b)=0,$$

and $p>0$ and $\sigma>0$

1. All the eigenvalues λ are real.
2. There exist an infinite number of eigenvalues:
$$\lambda_1<\lambda_2<\ldots<\lambda_n<\lambda_{n+1}<\ldots.$$
   a. There is a smallest eigenvalue, usually denoted λ_1.
   b. There is not a largest eigenvalue and $\lambda_n\to\infty$ as $n\to\infty$.
3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a<x<b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x)\sim\sum_{n=1}^{\infty}a_n\phi_n(x).$$
Furthermore, this infinite series converges to $[f(x+)+f(x-)]/2$ for $a<x<b$ (if the coefficients a_n are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
$$\int_a^b\phi_n(x)\phi_m(x)\sigma(x)\,dx=0 \quad \text{if } \lambda_n\neq\lambda_m.$$
6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:
$$\lambda=\frac{-p\phi\,d\phi/dx\big|_a^b+\int_a^b[p(d\phi/dx)^2-q\phi^2]\,dx}{\int_a^b\phi^2\sigma\,dx},$$
where the boundary conditions may somewhat simplify this expression.

$$\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)}_{L\phi} + q(x)\phi + \lambda\sigma(x)\phi = 0, \qquad a < x < b,$$

Thus I can write the PDE as $L\phi = -\lambda\sigma\phi$

looks like $Ax = \lambda x$ in linear algebra..

$$(f, Lg) = \int_a^b f(x)\,Lg(x)\,dx = \int_a^b f(x)\left[\frac{d}{dx}\left(p(x)\frac{dg(x)}{dx}\right) + q(x)g(x)\right]dx$$

compared with

okay

$$(Lf, g) = \int_a^b Lf(x)\,g(x)\,dx = \int_a^b \left[\frac{d}{dx}\left(p(x)\frac{df(x)}{dx}\right) + q(x)f(x)\right]g(x)\,dx.$$

Integration by parts

$$\int_a^b f(x)\frac{d}{dx}\left(p(x)\frac{dg(x)}{dx}\right)dx = \boxed{f(x)p(x)g'(x)\Big|_a^b} - \boxed{\int_a^b p(x)\frac{dg(x)}{dx}f'(x)\,dx}$$

$u = f(x)$ $\qquad du = f'(x)\,dx$

also

$dv = \frac{d}{dx}\left(p(x)\frac{dg(x)}{dx}\right)dx$ $\qquad v = p(x)\frac{dg(x)}{dx}$

same

$$\int_a^b \frac{d}{dx}\left(p(x)\frac{df(x)}{dx}\right)g(x)\,dx = \boxed{g(x)p(x)f'(x)\Big|_a^b} - \boxed{\int_a^b p(x)f'(x)g'(x)\,dx}$$

$u = g(x)$ $\qquad du = g'(x)$

$dv = \frac{d}{dx}\left(p(x)\frac{df(x)}{dx}\right)dx$ $\qquad v = p(x)\frac{df(x)}{dx}$

What's left are the boundary terms: Are these equal?

$$f(x)\,p(x)\,g'(x)\Big|_a^b \stackrel{?}{=} g(x)\,p(x)\,f'(x)\Big|_a^b$$

Boundary conditions

$$\boxed{\beta_2 f'(a) = -\beta_1 f(a)} \qquad \boxed{\beta_2 g'(a) = -\beta_1 g(a)}$$

$$\boxed{\beta_4 f'(b) = -\beta_3 f(b)} \qquad \boxed{\beta_4 g'(b) = -\beta_3 g(b)}$$

$$f(x)\,p(x)\,g'(x)\Big|_a^b = \boxed{f(b)\,p(b)\left(-\frac{\beta_3}{\beta_4} g(b)\right)} - \boxed{f(a)\,p(a)\left(-\frac{\beta_1}{\beta_2} g(a)\right)}$$

same $\updownarrow$ $\qquad$ same $\updownarrow$

$$g(x)\,p(x)\,f'(x)\Big|_a^b = \boxed{g(b)\,p(b)\left(-\frac{\beta_3}{\beta_4} f(b)\right)} - \boxed{g(a)\,p(a)\left(-\frac{\beta_1}{\beta_2} f(a)\right)}$$

5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$

recall $\sigma > 0$

$$L\phi = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi$$

$$L\phi = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi$$

$$\boxed{L\phi_n = -\lambda_n \sigma \phi_n}$$

so (λ_n, ϕ_n) are eigenvalue eigenfunction pairs of solutions to the ODE.

We know $L^\dagger = L$

thus $(L\phi_m, \phi_n) = (\phi_m, L\phi_n)$

Thus

$$(L\phi_m, \phi_n) = \int_a^b L\phi_m(x)\,\phi_n(x)\,dx = \int_a^b -\lambda_m \sigma(x)\,\phi_m(x)\,\phi_n(x)\,dx$$

$$= -\lambda_m \int_a^b \sigma(x)\,\phi_m(x)\,\phi_n(x)\,dx$$

$$(\phi_m, L\phi_n) = \int_a^b \phi_m(x)\,L\phi_n(x)\,dx = \int_a^b \phi_m(x)\left(-\lambda_n \sigma(x)\,\phi_n(x)\right)dx$$

$$= -\lambda_n \int_a^b \sigma(x)\,\phi_m(x)\,\phi_n(x)\,dx$$

implies

$$-\lambda_m \int_a^b \sigma(x)\,\phi_m(x)\,\phi_n(x)\,dx = -\lambda_n \int_a^b \sigma(x)\,\phi_m(x)\,\phi_n(x)\,dx$$

or

$$(\lambda_n - \lambda_m) \int_a^b \sigma(x)\,\phi_m(x)\,\phi_n(x)\,dx = 0$$

Since $\lambda_n \neq \lambda_m$ then $\lambda_n - \lambda_m \neq 0$. That implies

$$\int_a^b \sigma(x)\,\phi_m(x)\,\phi_n(x)\,dx = 0$$

Thus the eigenfunction are orthogonal with respect to the weight function σ,

$$(f,g)_\sigma = \int_a^b f(x)\,g(x)\,\sigma(x)\,dx$$

Now $(\phi_m, \phi_n)_\sigma = 0$ for $\lambda_m \neq \lambda_n$.

1. All the eigenvalues λ are real.

In linear algebra ... we know the eigenvalues of a symmetric matrix are real.

Let $Ax = \lambda x$. Suppose $\lambda \neq \bar{\lambda}$. Then what?

$$\left.\begin{aligned} \overline{Ax} &= \overline{\lambda x} \\ A\bar{x} &= \bar{\lambda}\bar{x} \end{aligned}\right\}$$ main idea...

$$\bar{x}\cdot Ax = \bar{x}\cdot\lambda x = \lambda\,\bar{x}\cdot x = \lambda\,\|x\|^2$$

$$\parallel$$

$$A^T\bar{x}\cdot x = A\bar{x}\cdot x = \bar{\lambda}\,\bar{x}\cdot x = \bar{\lambda}\,\|x\|^2$$

Conclusion $\lambda\cancel{\|x\|^2} = \bar{\lambda}\cancel{\|x\|^2}$ so $\lambda = \bar{\lambda}$.

This means λ is real

Similar proof for eigenvalues of eigenfunctions...

Suppose $L\phi = -\lambda\sigma\phi$

then $\overline{L\phi} = \overline{-\lambda\sigma\phi}$

$$\cancel{\bar{L}}\bar{\phi} = -\bar{\lambda}\cancel{\bar{\sigma}}\bar{\phi}$$

recall

$$L\phi_n = -\lambda_n\sigma\phi_n$$

$\sigma > 0$ so real

L is also real $L\phi = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi$ $p > 0$ so real and q is also real recall

where β_i are real. In addition, to be called regular, the coefficients p, q, and σ must be real and continuous everywhere (including the endpoints) and $p > 0$ and $\sigma > 0$ everywhere (also including the endpoints). For the regular Sturm–Liouville eigenvalue problem, many

Therefore

$$L\bar{\phi} = -\bar{\lambda}\sigma\bar{\phi}$$

Thus $\bar{\phi}$ is an eigenvector with eigenvalue $\bar{\lambda}$,

$$(\bar{\phi}, L\phi) = \int_a^b \bar{\phi}(x)\, L\phi(x)\, dx = \int_a^b \bar{\phi}(x)\left(-\lambda\sigma(x)\phi(x)\right) dx$$

$$\parallel$$

$$(L^{\dagger}\bar{\phi}, \phi) = (L\bar{\phi}, \phi) = \int_a^b \left(-\bar{\lambda}\sigma(x)\bar{\phi}(x)\right)\phi(x)\, dx$$

Therefore

$$\int_a^b \bar{\phi}(x)\left(-\lambda\sigma(x)\phi(x)\right) dx = \int_a^b \left(-\bar{\lambda}\sigma(x)\bar{\phi}(x)\right)\phi(x)\, dx$$

$$-\lambda\int_a^b \bar{\phi}(x)\,\sigma(x)\phi(x)\, dx = -\bar{\lambda}\int_a^b \sigma(x)\bar{\phi}(x)\phi(x)\, dx$$

Note:

$$\int_a^b \bar{\phi}(x)\,\sigma(x)\phi(x)\, dx = \int_a^b \sigma(x)|\phi(x)|^2\, dx$$

non-negative.

positive since $\sigma > 0$.

also know $\phi(x) \neq 0$.

Since ϕ is an eigenfunction it is non-zero

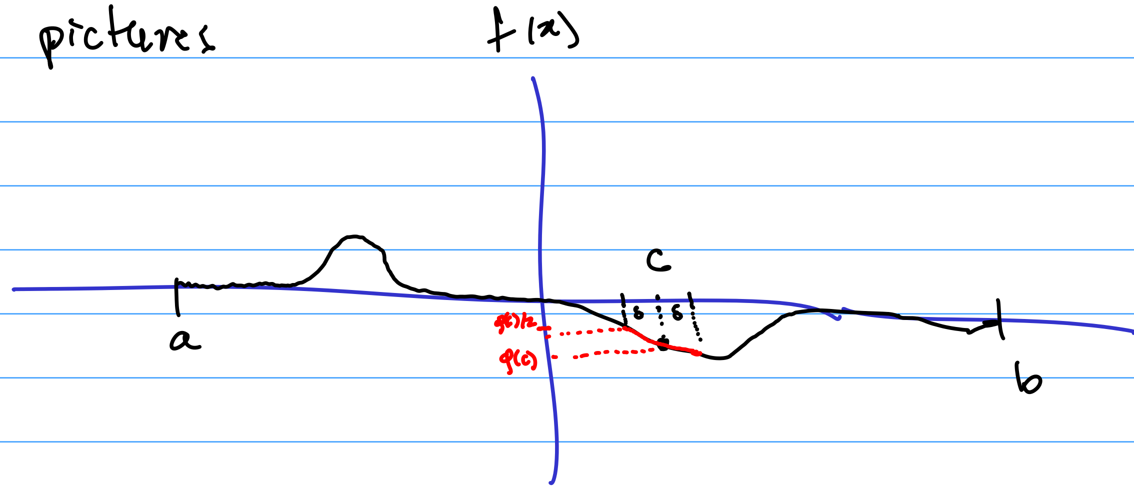
Then there is a point $c \in [a,b]$ such that $\phi(c) \neq 0$.

since it's solution to an ODE then it is differentiable and so continuous. Thus $f(x) = \sigma(x)|\phi(x)|^2$ is continous.

There is a $\delta > 0$ such that

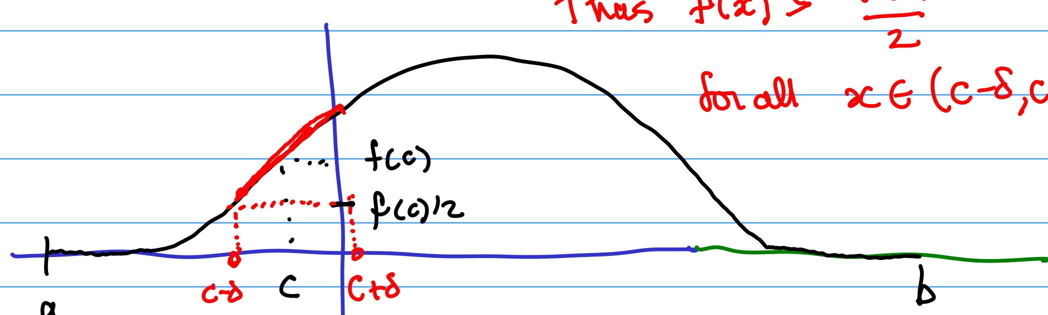
$$|f(x)| > \frac{|f(c)|}{2} \text{ for all } x \in (c-\delta, c+\delta) \cap [a,b].$$

In pictures



Thus $f(x) > \frac{f(c)}{2}$

for all $x \in (c-\delta, c+\delta)$



Conclusion $\displaystyle\int_a^b \sigma(x)|\phi(x)|^2\,dx \geq \int_{c-\delta}^{c+\delta} \frac{|f(x)|}{2}\,dx > 0$

$$-\lambda \cancel{\int_a^b \bar{\varphi}(x)\,\sigma(x)\,\varphi(x)\,dx} = -\bar{\lambda} \cancel{\int_a^b \sigma(x)\,\bar{\varphi}(x)\,\varphi(x)\,dx}$$

since integral is
non zero

So $\lambda = \bar{\lambda}$ so eigenvalues are real.