Method of Characteristics

Wave equation on $\mathbb{R}$.

unbounded domain so no boundary conditions

PDE $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{for } x \in \mathbb{R} \text{ and } t \geq 0$$

I.C. $u(x,0) = f(x)$ ← initial displacement

$u_t(x,0) = g(x)$ ← initial velocity...

Solve by factoring the differential operator into 1st order PDEs. Then solve the 1st order PDEs by characteristics. Put the pieces back together.

$$\underbrace{\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)}_{\text{difference of squares...}} u = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\underbrace{\boxed{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u}}_{v} = 0 \quad \text{or} \quad \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\underbrace{\boxed{\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u}}_{w} = 0$$

Make a change of variables to see the first order PDEs.

Let $\quad v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} \quad$ and $\quad w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$

Thus, we get these 1st order PDEs...

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)v = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)w = 0$$

Want to solve $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)v = 0$ using the method of characteristics...

Idea let $x = x(s)$ and $t = t(s)$ and see what happens when we differentiate v with respect to s.

$$\frac{d}{ds}v(x(s), t(s)) = \boxed{\frac{\partial v}{\partial x}}\, x'(s) + \boxed{\frac{\partial v}{\partial t}}\, t'(s)$$

Compare this to the PDE.

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)v = 1\,\boxed{\frac{\partial v}{\partial t}} - c\,\boxed{\frac{\partial v}{\partial x}} = 0$$

I have two ODEs..

$$x'(s) = -c \qquad t'(s) = 1$$

$$x(s) = -cs + x(0) \qquad t(s) = s + \cancel{t(0)} = s$$

Thus, setting

$$v = v(-cs + x(0), s)$$

$$\frac{dv}{ds} = \frac{\partial v}{\partial x}x'(s) + \frac{\partial v}{\partial t}t'(s) = 1\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$$

We have the ODE

$$\frac{dv}{ds} = 0 \qquad v(s) = v\Big|_{s=0} = \text{constant.}$$

Let solve

PDE $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0$ for $x \in \mathbb{R},\ t \geq 0$

I.C. $v(x,0) = \alpha(x)$ for $x \in \mathbb{R}$

Thus

$$v(x(s), t(s)) = v(-cs + x(0), s) = v(-cs + x(0), s)\Big|_{s=0}$$

$$= v(x(0), 0) = \alpha(x(0))$$

Again

$$v(\underbrace{-cs + x(0)}_{x}, \underbrace{s}_{t}) = \alpha(x(0))$$

$x = -cs + x(0)$ $\qquad$ $x(0) = x + cs$

$s = t$ $\qquad$ $t = s$.

Thus

$$v(x,t) = \alpha(x + cs) = \alpha(x + ct)$$

or $\quad v(x,t) = \alpha(x + ct)$,

is the solution to

PDE $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0$ for $x \in \mathbb{R},\ t \geq 0$

I.C. $v(x,0) = \alpha(x)$ for $x \in \mathbb{R}$

Solve:

PDE $\quad 1 \cdot \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \quad$ for $\quad x \in \mathbb{R},\ t \geq 0$

I.C. $\quad w(x,0) = \beta(x) \quad$ for $\quad x \in \mathbb{R}$

Shortcut let t be the parameter...

$$w = w(x(t), t)$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} x'(t) + \frac{\partial w}{\partial t}$$

characteristic

Get the ODE $x'(t) = c$ so $\boxed{x(t) = ct + x(0)}$

Then $\boxed{\frac{d}{dt} w(ct + x(0), t) = 1 \cdot \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0}$ what the PDE becomes along the characteristic direction.

so $w(ct + x(0), t) = \text{const} = w(ct + x(0), t)\Big|_{t=0} = w(x(0), 0) = \beta(x(0))$

Thus

$$\underbrace{w(ct + x(0)}_{x = ct + x(0)}, t) = \beta(x(0))$$

so $x(0) = x - ct$

implies $\quad w(x,t) = \beta(x - ct)$

is a solution to

PDE $\quad 1 \cdot \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \quad$ for $\quad x \in \mathbb{R},\ t \geq 0$

I.C. $\quad w(x,0) = \beta(x) \quad$ for $\quad x \in \mathbb{R}$

Put the solutions back together to solve.

PDE $\qquad \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \qquad$ for $x \in \mathbb{R}$ and $t \geq 0$

I.C. $\qquad u(x,0) = f(x) \leftarrow$ initial displacement

$\qquad u_t(x,0) = g(x) \leftarrow$ initial velocity ...

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) w = 0$$

$$v(x,0) = \alpha(x) \qquad\qquad w(x,0) = \beta(x)$$

where $v = \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x}$ and $w = \dfrac{\partial u}{\partial t} - c\dfrac{\partial u}{\partial x}$

$$v(x,t) = \alpha(x+ct), \qquad w(x,t) = \beta(x-ct)$$

Therefore

$$\alpha(x+ct) = \frac{\partial u}{\partial t} + c\cancel{\frac{\partial u}{\partial x}} \qquad\qquad \alpha(x+ct) = \cancel{\frac{\partial u}{\partial t}} + c\frac{\partial u}{\partial x}$$

$$\beta(x-ct) = \frac{\partial u}{\partial t} - c\cancel{\frac{\partial u}{\partial x}} \qquad\qquad -\left(\beta(x-ct) = \cancel{\frac{\partial u}{\partial t}} - c\frac{\partial u}{\partial x}\right)$$

$$\alpha(x+ct) + \beta(x-ct) = 2\frac{\partial u}{\partial t} \qquad\qquad \alpha(x+ct) - \beta(x-ct) = 2c\frac{\partial u}{\partial x}$$

Question... what is $u(x,t)$?

$$\int_0^x \big(\alpha(x+ct) - \beta(x-ct)\big)\,dx = \int_0^x 2c\frac{\partial u}{\partial x}\,dx = 2c\,u(x,t) - 2c\,u(0,t)$$

Question ... what is $u(0,t)$?

$$\int_0^t \left(\alpha(ct)+\beta(-ct)\right)dt = \int_0^t 2u_t(0,s)\,ds = 2u(0,t) - 2u(0,0)$$

Add these two equations together

$$\int_0^x \left(\alpha(x+ct)-\beta(x-ct)\right)dx = \int_0^x 2c\frac{\partial u}{\partial x}\,dx = 2c\,u(x,t) - 2c\,u(0,t)$$

$$c\int_0^t \left(\alpha(ct)+\beta(-ct)\right)dt = c\int_0^t 2u_t(0,s)\,ds = 2c\,u(0,t) - 2c\,u(0,0)$$

$$\int_0^x \left(\alpha(x+ct)-\beta(x-ct)\right)dx + c\int_0^t \left(\alpha(ct)+\beta(-ct)\right)dt = 2c\,u(x,t) - 2c\,u(0,0)$$

$$u(x,t) = \frac{1}{2c}\left(\int_0^x \left(\alpha(x+ct)-\beta(x-ct)\right)dx + c\int_0^t \left(\alpha(ct)+\beta(-ct)\right)dt\right) + u(0,0).$$

Solve for α and β so that I.C. are satisfied

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$