$$u(x,t) = \frac{1}{2c}\left(\int_0^x \big(\alpha(x+ct) - \beta(x-ct)\big)\,dx + c\int_0^t \big(\alpha(ct)+\beta(-ct)\big)\,dt\right) + u(0,0).$$

$$\int_0^x \alpha(x+ct)\,dx = \int_{ct}^{x+ct} \alpha(z)\,dz \qquad \int_0^x \beta(x-ct)\,dx = \int_{-ct}^{x-ct} \beta(z)\,dz$$

$z = x+ct$, $dz = dx$ $\qquad$ $z = x-ct$, $dz = dx$

$$c\int_0^t \alpha(ct)\,dt = \int_0^{ct} \alpha(z)\,dz \qquad c\int_0^t \beta(-ct)\,dt = -\int_0^{-ct} \beta(z)\,dz$$

$z = ct$, $dz = c\,dt$ $\qquad$ $z = -ct$, $dz = -c\,dt$

$$u(x,t) = u(0,0) + \frac{1}{2c}\left(\int_{ct}^{x+ct} \alpha(z)\,dz - \int_{-ct}^{x-ct} \beta(z)\,dz + \int_0^{ct} \alpha(z)\,dz - \int_0^{-ct} \beta(z)\,dz\right)$$

Therefore

$$u(x,t) = \underbrace{u(0,0)}_{\text{const}} + \underbrace{\frac{1}{2c}\int_0^{x+ct} \alpha(z)\,dz}_{\text{this is a function of } x+ct} \underbrace{- \frac{1}{2c}\int_0^{x-ct} \beta(z)\,dz}_{\text{this is a function of } x-ct}$$

Thus

$$u(x,t) = \boxed{F(x-ct)} + \boxed{G(x+ct)}$$

So solving for α and β is the same as solving for F and G.

We want

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

$u(0,0) = f(0)$

Thus

$$u(x,0) = F(x-\cancel{ct}) + G(x+\cancel{ct})\Big|_{t=0} = f(x)$$

or

$$F(x) + G(x) = f(x)$$

$$u(x,t) = F(x-ct) + G(x+ct)$$
$$u_t(x,t) = -cF'(x-ct) + cG'(x+ct)$$

$$u_t(x,0) = -cF'(x) + cG'(x) = g(x)$$

Thus we need to solve

$$F + G = f$$
$$-\cancel{cF'} + cG' = g \qquad \text{for } F \text{ and } G.$$
$$\cancel{cF'} + cG' = cf'$$

$$2cG' = cf' + g$$
$$2cF' = cf' - g$$

integrate to obtain G and F.

$$\int_0^x 2c\,F'(s)\,ds = \int_0^x c f'(s)\,ds - \int_0^x g(s)\,ds$$

$$2c\left(F(x) - F(0)\right) = c\left(f(x) - f(0)\right) - \int_0^x g(s)\,ds$$

$$F(x) = \boxed{F(0)} + \frac{1}{2}f(x) - \frac{1}{2}f(0) - \frac{1}{2c}\int_0^x g(s)\,ds.$$

?

$$\int_0^x 2c\,G'(s)\,ds = \int_0^x c f'(s)\,ds + \int_0^x g(s)\,ds$$

$$2c\left(G(x) - G(0)\right) = c\left(f(x) - f(0)\right) + \int_0^x g(s)\,ds$$

?

$$G(x) = \boxed{G(0)} + \frac{1}{2}f(x) - \frac{1}{2}f(0) + \frac{1}{2c}\int_0^x g(s)\,ds.$$

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$= \boxed{F(0)} + \frac{1}{2}f(x-ct) - \frac{1}{2}f(0) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds.$$

$$+ \boxed{G(0)} + \frac{1}{2}f(x+ct) - \frac{1}{2}f(0) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds$$

$$u(0,0) = F(0) + \frac{1}{2}f(0) - \frac{1}{2}f(0) - \frac{1}{2c}\int_0^0 g(s)\,ds$$

$$+ G(0) + \frac{1}{2}f(0) - \frac{1}{2}f(0) + \frac{1}{2c}\int_0^0 g(s)\,ds$$

Thus $u(0,0) = F(0) + G(0)$

$$u(x,t) = \cancel{u(0,0)} - \cancel{f(0)} + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds$$

Since $u(0,0) = f(0)$

Therefore.

$$u(x,t) = \boxed{\frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds} + \boxed{\frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds}$$

$$G(x) = \boxed{\frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\overline{x})\ d\overline{x}}$$

$$F(x) = \boxed{\frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\overline{x})\ d\overline{x}}$$

In summary, we solved

PDE $\quad \dfrac{\partial^2 u}{\partial t^2} = c\dfrac{\partial^2 u}{\partial x^2}$ on $x \in \mathbb{R}$ and $t \geq 0$

IC $\quad u(x,0) = f(x)$ for $x \in \mathbb{R}$

$\quad u_t(x,0) = g(x)$

Solution:

$$u(x,t) = \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds + \boxed{\frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds}$$

Focus is the part where we solve a 1^{st} order PDE by introducing a parameterization

$$x = x(s) \qquad t = t(s)$$

and then solving ODEs

$$x'(s) = \text{something}$$
$$t'(s) = \text{something.}$$

And the somethings are design so that

$$\frac{du}{ds} = \text{something}$$

is an ODE representing the original PDE solved along the parameterized curve $(x(s), t(s))$.

Since not using superposition, this idea can work for non-linear as well as linear PDEs.

Solve using the method of characteristics.

(a) $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x}$ with $w(x,0) = f(x)$

***(b)** $\frac{\partial w}{\partial t} + x\frac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

(c) $\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

***(d)** $\frac{\partial w}{\partial t} + 3t\frac{\partial w}{\partial x} = w$ with $w(x,0) = f(x)$