Use method of characteristics to solve.

(c) $1\cdot\dfrac{\partial w}{\partial t} + t\dfrac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

Since there is 1 here lets use t as the parameter...

$$x = x(t)$$

$$\frac{d}{dt} w(x(t),t) = \frac{\partial w}{\partial t} + x'(t)\frac{\partial w}{\partial x}$$

Use this identification to obtain 2 ODES.

$$x'(t) = t \quad \text{and} \quad \frac{d}{dt} w(x(t),t) = 1$$

const.

$$x(t) = \frac{1}{2}t^2 + x_0 \qquad w(x(t),t) = t + c.$$

Therefore $\quad w\left(\frac{1}{2}t^2 + x_0, t\right) = t + c$

$$w(x_0, 0) = 0 + c \qquad \text{so} \quad c = w(x_0,0) = f(x_0)$$

Thus

$$w(\underbrace{\tfrac{1}{2}t^2 + x_0}_{x}, t) = t + f(x_0)$$

Change the variables $\quad x = \frac{1}{2}t^2 + x_0, \quad x_0 = x - \frac{1}{2}t^2$

$$\boxed{w(x,t) = t + f\left(x - \tfrac{1}{2}t^2\right)}$$

answer...

Did it work? Check the answer?

(c) $\dfrac{\partial w}{\partial t} + t\dfrac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

Solution. $w(x,t) = t + f(x - \frac{1}{2}t^2)$

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left(t + f(x - \tfrac{1}{2}t^2)\right) = 1 + f'(x - \tfrac{1}{2}t^2)(-t)$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(t + f(x - \tfrac{1}{2}t^2)\right) = 0 + f'(x - \tfrac{1}{2}t^2)$$

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1 - t\,\cancel{f'(x - \tfrac{1}{2}t^2)} + t\left(\cancel{f'(x - \tfrac{1}{2}t^2)}\right) = 1$$

so this satisfies the PDE

What about initial condition?

$$w(x,0) = 0 + f(x - \tfrac{1}{2}0^2) = f(x)$$

***12.2.6.** Consider (if necessary, see Section 12.6):

$$1. \quad \frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0 \quad \text{with} \quad u(x,0) = f(x).$$

Show that the characteristics are straight lines.

Try method of characteristics. Try a parameterization with respect to time...

$$x = x(t)$$

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial t} + x'(t)\frac{\partial u}{\partial x}$$

With this identification we obtain the ODEs

$$x'(t) = 2u(x(t),t) \qquad \frac{du(x(t),t)}{dt} = 0$$

Note the ODE's are coupled, but if I solve them in the correct order, it's still possible

$$\frac{d\,u(x(t),t)}{dt} = 0$$

$u(x(t),t) = \text{const}$

$u(x(0),0) = f(x(0))$

$u(x(t),t) = f(x(0))$

Now plug into the other ODE

$x'(t) = 2u(x(t),t) = 2f(x(0))$

$x(t) = 2tf(x(0)) + x(0)$

The characteristics are $(x(t),t)$

$= (2tf(x_0)+x_0, t)$

this is a straight line.

Therefore, along the characteristics

$$u(\underbrace{2tf(x_0)+x_0}_{x}, t) = f(x_0)$$

Let $x = 2tf(x_0)+x_0$ ..

How to solve for x_0?

need to know what f is to solve for x_0 and maybe this equation isn't even invertible ... note the invertibility may also depend on what t is.

The reason this turned out different is because the PDE

1. $$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

is nonlinear ...

this is a quadratic term in u ...

The solution may develop a shock, or discontinuity as t increases...

$$\boxed{\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = Q,}$$

where c and Q may be functions of x, t, and ρ.

important to us when the coefficient c depend

Note c and Q don't have any derivatives in them...

1. $$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

plays the role of ρ.

my c is a function of u

Note $$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 0$$

not a quasilinear term.

Try solving this...

(b) $$\frac{\partial \rho}{\partial t} + \rho\frac{\partial \rho}{\partial x} = t \qquad \rho(x, 0) = f(x):$$

Method of characteristics.

$x = x(t)$

$$\frac{d}{dt}\rho(x(t), t) = \frac{\partial \rho}{\partial t} + x'(t)\frac{\partial \rho}{\partial x}$$

We get two ODEs

$$x'(t) = \rho(x(t), t) \qquad \frac{d}{dt}\rho(x(t), t) = t$$

this one first

Thus

$$\frac{d}{dt}\rho(x(t), t) = t \qquad \rho(x(t), t) = \frac{1}{2}t^2 + \rho(x(0), 0)$$

plug in here

$$\rho(x(t), t) = \frac{1}{2}t^2 + f(x_0)$$

const.

$$x'(t) = \frac{1}{2}t^2 + f(x_0) \qquad x(t) = \frac{1}{6}t^3 + t f(x_0) + x_0$$

Therefore along the characteristics ...

$$\rho\left(\frac{1}{6}t^3 + t f(x_0) + x_0, t\right) = \frac{1}{2}t^2 + f(x_0)$$

Let $x = \frac{1}{6}t^3 + t f(x_0) + x_0$

and try to solve for x_0 ... if this is possible will depend on t and f.