Consider

1. $$\frac{\partial \rho}{\partial t} + t^2 \rho \frac{\partial \rho}{\partial x} = -\rho \quad \text{for } x \in \mathbb{R} \text{ and } t \geq 0.$$

$$\rho(x,0) = f(x)$$

Use the method of characteristics. Since no coefficient in front of $\partial\rho/\partial t$ use t as the parameter.

$$x = x(t)$$

$$\frac{d}{dt}\rho(x(t),t) = \frac{\partial \rho}{\partial t} + x'(t)\frac{\partial \rho}{\partial x}$$

Thus we get two ODEs...

$$x'(t) = t^2 \rho(x(t),t) \qquad \frac{d}{dt}\rho(x(t),t) = -\rho(x(t),t)$$

Need to know what ρ is to solve for x.

$$\frac{d}{dt}\rho = -\rho \qquad \rho = Ce^{-t}$$

$$\rho(x(t),t) = Ce^{-t}$$

$$\rho(x(0),0) = Ce^{-0} = C = f(x(0))$$

$x_0 = x(0)$

$$\boxed{\rho(x(t),t)} = f(x_0)e^{-t}$$

Substitute ρ into the first ODE.

$$x'(t) = t^2 f(x_0) e^{-t}$$

$$\boxed{x(t) = x_0 + f(x_0)\int_0^t s^2 e^{-s}\, ds}$$

$t=0$

$$x(0) = x_0 + f(x_0)\int_0^0 s^2 e^{-s} ds = x_0$$

guess and check

$$\int t^2 e^{-t}\,dt = (at^2+bt+c)e^{-t}$$

$$\frac{d}{dt}\int t^2 e^{-t}\,dt = \frac{d}{dt}(at^2+bt+c)e^{-t}$$

$$t^2e^{-t} = (-at^2-bt-c)e^{-t} + (2at+b)e^{-t}$$

$$t^2e^{-t} = (-at^2+(2a-b)t+(b-c))\,e^{-t}$$

$-a=1 \qquad a=-1$

$2a-b=0 \qquad b=2a=-2$

$b-c=0 \qquad c=b=-2$

$$\int t^2e^{-t}\,ds = -(t^2+2t+2)e^{-t} + \text{Const.}$$

$$\int_0^t s^2e^{-s}\,ds = -(t^2+2t+2)e^{-t}+2$$

Therefore

$$x(t) = x_0 + f(x_0)\int_0^t s^2e^{-s}\,ds$$

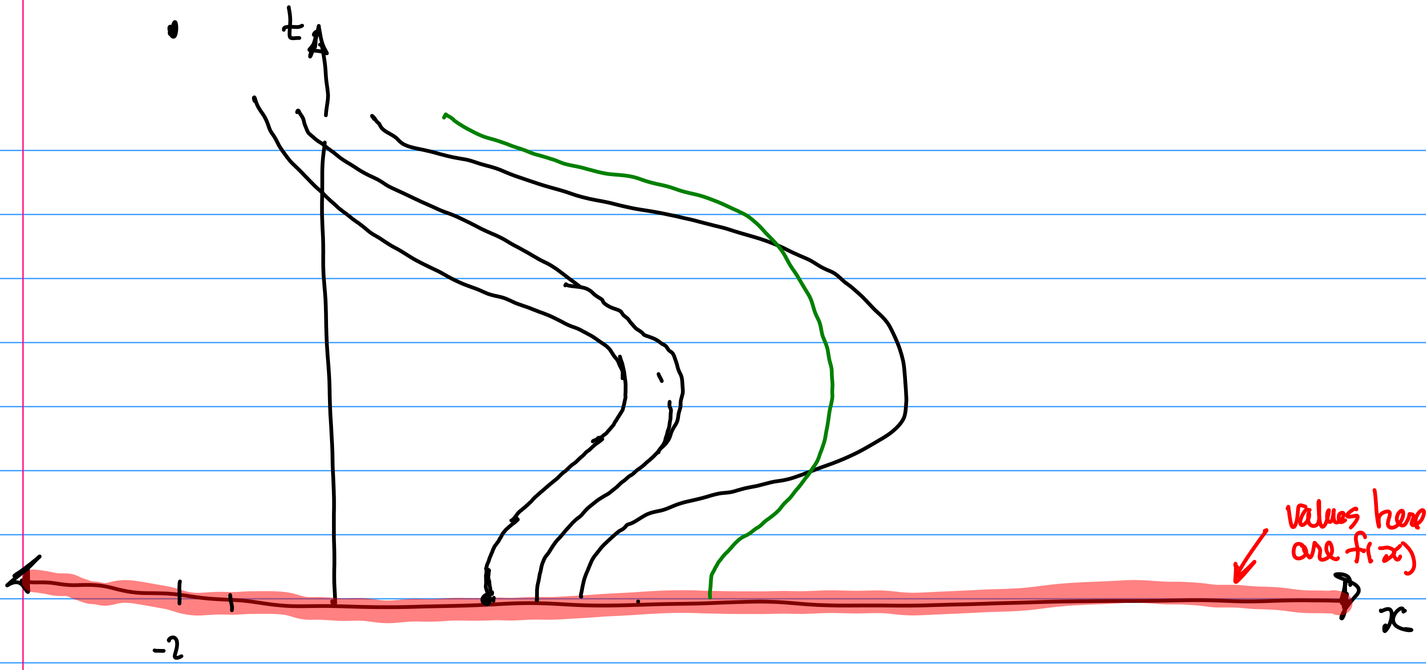
$$x(t) = x_0 - f(x_0)\left((t^2+2t+2)e^{-t}-2\right)$$

implies

Answer

$$\boxed{\rho\left(x_0 - f(x_0)\left((t^2+2t+2)e^{-t}-2\right),\, t\right) = f(x_0)e^{-t}}$$

implicit along characteristics...



could be shocks depending on the initial cond...

$$x(t) = x_0 - f(x_0)\left((t^2+2t+2)e^{-t} - 2\right)$$

$$x'(t) = t^2 f(x_0) e^{-t}$$

$$x'(0) = 0$$

12.6.9. Determine a parametric representation of the solution satisfying $\rho(x,0) = f(x)$:

*(a) $\dfrac{\partial \rho}{\partial t} - \rho^2 \dfrac{\partial \rho}{\partial x} = 3\rho$

(b) $\dfrac{\partial \rho}{\partial t} + \rho \dfrac{\partial \rho}{\partial x} = t$

*(c) $\boxed{\dfrac{\partial \rho}{\partial t} + t^2 \rho \dfrac{\partial \rho}{\partial x} = -\rho}$

(d) $\dfrac{\partial \rho}{\partial t} + \rho \dfrac{\partial \rho}{\partial x} = -x\rho$

answer...

$$\rho\left(x_0 - f(x_0)\left((t^2+2t+2)e^{-t} - 2\right), t\right) = f(x_0) e^{-t}$$

like ~~fan out~~? or shock?

Solve

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0 \qquad \rho(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$$

$x = x(t)$

$$\frac{d}{dt}\rho(x(t),t) = \frac{\partial \rho}{\partial t} + x'(t)\frac{\partial \rho}{\partial x}$$

ODEs

$$x'(t) = \rho^2 = \rho(x_0,0)^2 \qquad \frac{d\rho}{dt} = 0$$

$$x(t) = \rho(x_0,0)^2 t + x_0 \qquad \rho(x(t),t) = \text{const} = \rho(x_0,0)$$

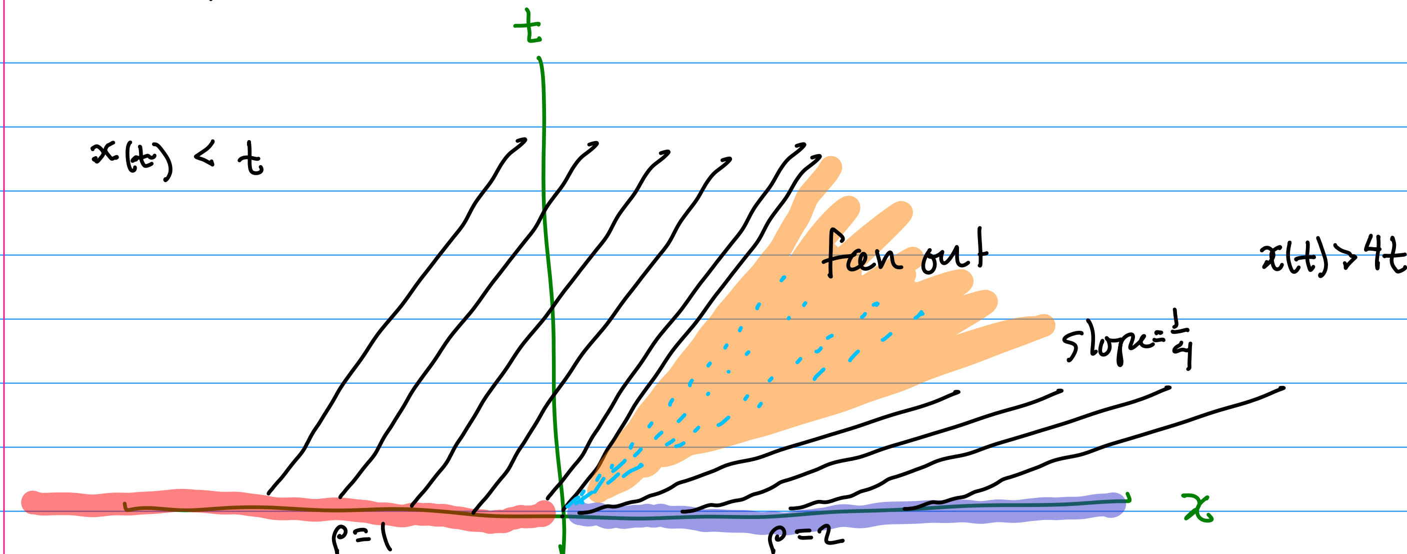
$$= \begin{cases} 1 & \text{for } x_0 < 0 \\ 2 & \text{for } x_0 > 0 \end{cases}$$

along characteristics...

$$\rho\left(\rho(x_0,0)^2 t + x_0, t\right) = \rho(x_0,0)$$

Do characteristics intersect or fan out?

$$x(t) = \rho(x_0,0)^2 t + x_0$$



$$\rho\left(\rho(x_0,0)^2 t + x_0, t\right) = \rho(x_0,0)$$

$$\rho(x,t) = \begin{cases} 1 & \text{if } x < t \\ \sqrt{\frac{x}{t}} & t < x < 4t \\ 2 & \text{if } x > 4t \end{cases}$$

$x = \rho^2 t$ $\qquad\qquad$ $\rho \in [1,2]$

$\rho^2 = \frac{x}{t}$, $\quad \rho = \sqrt{\frac{x}{t}}$

This problem was similar to

1.6.7(a) $\dfrac{\partial \rho}{\partial t} + \rho^2 \dfrac{\partial \rho}{\partial x} = 0, \qquad \rho(x,0) = \begin{cases} 3 & x < 0 \\ 4 & x > 0 \end{cases}$