From last time

$$\int_0^L \sin\frac{n\pi}{L}x \, \sin\frac{m\pi}{L}x \, dx$$

$$= \frac{1}{2}\int_0^L \left[\cos\frac{(n-m)\pi}{L}x - \cos\frac{(n+m)\pi}{L}x\right] dx$$

Case $n = m$ and $n > 0$, $m > 0$

$$\frac{1}{2}\int_0^L \left(1 - \cos\frac{2\pi n}{L}x\right) dx$$

$$= \frac{L}{2} - \frac{L}{2\pi n}\sin\frac{2\pi n x}{L}\Big|_0^L = \frac{L}{2}$$

Case $n \neq m$ and $n > 0$, $m > 0$.

$$\int_0^L \sin\frac{n\pi}{L}x \, \sin\frac{m\pi}{L}x \, dx =$$

$$= \frac{1}{2}\int_0^L \left(\cos\frac{(n-m)\pi}{L}x - \cos\frac{(n+m)\pi}{L}x\right) dx$$

both $n-m$ and $n+m$ are non-zero.

$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi}{L}x - \frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi}{L}x\right]_0^L = 0$$

~~Now solve for Bn's using the above orthogonality~~

Try to solve for B_n such that $u(x,0) = f(x)$

$$\sum_{n \in \mathbb{N}} B_n \left(\sin\frac{n\pi}{L}x\right) = f(x)$$

First multiply by $\sin\frac{m\pi}{L}x$

$$\sum_{n\in\mathbb{N}} B_n\left(\sin\frac{n\pi}{L}x\right)\left(\sin\frac{m\pi}{L}x\right) = f(x)\left(\sin\frac{m\pi}{L}x\right)$$

Then integrate

$$\int_0^L \sum_{n\in\mathbb{N}} B_n\left(\sin\frac{n\pi}{L}x\right)\left(\sin\frac{m\pi}{L}x\right)dx = \int_0^L f(x)\left(\sin\frac{m\pi}{L}x\right)dx$$

switch the order...

Dangerous mathematically because $\int$ is a limit and the sum is also a limit. We'll discuss more in the next chapter when this switching actually works...

$$\sum_{n=1}^{\infty} B_n \underbrace{\int_0^L \left(\sin\frac{n\pi}{L}x\right)\left(\sin\frac{m\pi}{L}x\right)dx}_{\text{use orthogonality here}} = \int_0^L f(x)\left(\sin\frac{m\pi}{L}x\right)dx$$

$$\int_0^L \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x = \begin{cases} \frac{L}{2} & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$$

only

$$B_m \frac{L}{2} = \int_0^L f(x)\left(\sin\frac{m\pi}{L}x\right)dx$$

$$\boxed{B_m = \frac{2}{L}\int_0^L f(x)\left(\sin\frac{m\pi}{L}x\right)dx}$$

Solution

Assuming all this works ... i.e. no problems occur when switching the limits ..

Then

$$u(x,t) = \sum_{n=1}^{\infty} B_n \left(\sin \frac{n\pi}{L} x\right) e^{-k\frac{n^2\pi^2}{L^2}t}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x) \left(\sin \frac{n\pi}{L} x\right) dx$$

Is the solution to the PDE

Heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{on } x \in [0, L] \text{ and } t \geq 0$$

Homogeneous Heat bath boundary

$$u(0,t) = 0 \qquad u(L,t) = 0 \qquad t \geq 0$$

Initial condition

$$u(x,0) = f(x) \quad \text{for} \quad x \in [0,L]$$

What about insulating boundary condition?

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{on } x \in [0, L] \text{ and } t \geq 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0 \qquad t \geq 0$$

$$u(x,0) = f(x) \quad \text{for} \quad x \in [0,L]$$

Separation of variables: look for a solution that is a superposition of

$$u(x,t) = \sum_n B_n f_n(x) G_n(t)$$

where $u_n(x,t) = \varphi_n(x)\, G_n(t)$ are solutions to the homogeneous boundary and the B_n's are chosen to satisfy the initial condition.

Let $u(x,t) = \varphi(x) G(t)$ (drop the subscripts for convenience)

Plug it in...

$$\frac{\partial}{\partial t}\left(\varphi(x) G(t)\right) = k \frac{\partial^2}{\partial x^2}\left(\varphi(x) G(t)\right)$$

$$\varphi(x)\, G'(t) = k\, G(t)\, \varphi''(x)$$

$$\frac{\varphi(x)}{\varphi''(x)} = \frac{k\,G(t)}{G'(t)} = \beta$$ ← would work just as well

or maybe like last time

$$\frac{G'(t)}{k\,G(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$ ← How it's done in the book, so that $\lambda > 0$ corresponds to the non-trivial solutions..

Thus

$$G'(t) = -\lambda k\, G(t) \quad \text{and} \quad \varphi''(x) = -\lambda \varphi(x)$$

and boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0 \qquad t \geq 0$$

$$\cancel{G(t)}\,\varphi'(0) = 0 \qquad \cancel{G(t)}\,\varphi'(L) = 0$$

since $G(t) = 0$ for all time would be the zero solution...

ODE:

$\varphi''(x) = -\lambda\varphi(x)$ s.t. $\varphi'(0) = 0$ and $\varphi'(L) = 0$

General solution of the ODE

$$\varphi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$$

Now satisfy the boundary conditions

Case $\lambda = 0$ $\quad \varphi(x) = c_1 + c_2 x$ and

$\varphi'(L) = 0$ $\quad$ so $\varphi(x) = c_1$

Case $\lambda > 0$

$\varphi'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$

$\varphi'(0) = c_2\sqrt{\lambda} = 0$ $\qquad c_2 = 0$ or $\lambda = 0$

Thus $c_2 = 0$ and $\varphi(x) = c_1 \cos\sqrt{\lambda}\, x$

$\varphi'(L) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, L = 0$ $\quad$ so $\sqrt{\lambda}\, L = \pi n$

$\sqrt{\lambda} = \frac{n\pi}{L}$

$\lambda = \frac{n^2\pi^2}{L^2}$

$\varphi(x) = c_1 \cos\frac{n\pi}{L} x$

Case $\lambda < 0$.

at home check if there are any non-zero solutions in this case. (NO)

Thus $\varphi_n(x) = C_n \cos \frac{n\pi}{L} x$ for $n = 0, 1, 2, \ldots$

Now find the $G_n(t)$ and then solve for the constants B_n.