Separation of variables

$$\boxed{u(x,t) = \phi(x)G(t),} \qquad (2.3.4)$$

substitute

Heat eq

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial}{\partial t}\left(\varphi(x)G(t)\right) = k\frac{\partial^2}{\partial x^2}\left(\varphi(x)G(t)\right)$$

$$\varphi(x)\frac{\partial}{\partial t}G(t) = k\,G(t)\frac{\partial^2}{\partial x^2}\varphi(x)$$

Thus, $\varphi G' = k G \varphi''$

Separation of variables means put all functions of t on one side and all functions of x on the other.

$$\frac{G'(t)}{k\,G(t)} = \frac{\varphi''(x)}{\varphi(x)} = \text{constant that depends neither on } x \text{ or } t. = -\lambda$$

only a function of t

only a function of x

$$\frac{G'(t)}{k\,G(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

Therefore $G'(t) = -\lambda k G(t)$ and $\varphi''(x) = -\lambda\varphi(x)$

Boundary conditions: $u(0,t)=0 \quad u(L,t)=0$

homogeneous boundary conditions

Heat equation: $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$

Suppose u_1 is a solution that satisfies the PDE and the boundary condions

and u_2 is another that satisfies the PDE and the boundary condions

Since the PDE is linear and the boundary cond. are homogeneous then

$$u = c_1 u_1 + c_2 u_2$$

is also a solution.

Separation of variables

$$u(x,t) = \phi(x)G(t)$$

Then u satisfies the heat equation when

$$G'(t) = -\lambda k G(t) \quad \text{and} \quad \phi''(x) = -\lambda\phi(x)$$

Boundary condion

$u(0,t) = 0 = \underbrace{\phi(0)G(t)}_{\text{means } \phi(0)=0}$ for all t

$u(L,t) = 0 = \phi(L)G(t)$ for all t

means $\varphi(L)=0$

Note, it may not be necessary for
$\varphi(0)=0$ and $\varphi(L)=0$
but this is sufficient.

Therefore we have the ODE.

$$\varphi''(x) = -\lambda\varphi(x)$$

$$\varphi(0)=0 \quad \text{and} \quad \varphi(L)=0$$

Solve it:

Case $\lambda=0$

Then $\varphi''(x)=0$ so $\varphi(x) = ax+b$

$\varphi(0)=0$ means $a\cdot 0+b=b=0$ so $b=0$

$\varphi(L)=0$ means $a\cdot L=0$ so $a=0$

thus $\varphi(x)=0$.

And $u(x,t)=\varphi(x)G(t)=0$ and using this solution by adding it in a superposition is not usefull

Case $\lambda<0$

Then $\varphi''(x)=-\lambda\varphi(x)$ then $-\lambda>0$.

$\varphi''+\lambda\varphi=0$ substitute $\varphi=e^{rx}$

$\varphi'=re^{rx}$

$\varphi''=r^2e^{rx}$

$$r^2\cancel{e^{rx}}+\lambda\cancel{e^{rx}}=0$$

$$r^2+\lambda=0$$ ← characteristic eq.

$$r^2=\pm\sqrt{-\lambda}$$

since $-\lambda>0$

General solution to the ODE is

$$\varphi(x) = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$$

Boundary condition:

$\varphi(0) = 0$ and $\varphi(L) = 0$

$\varphi(0) = c_1 + c_2 = 0$ so $c_1 = -c_2$

$$\varphi(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L} = 0$$

$$= c_1 e^{\sqrt{-\lambda}L} - c_1 e^{-\sqrt{-\lambda}L} = 0$$

$$= c_1 \big(\underbrace{e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}} \big) = 0$$

since $\lambda \neq 0$ then this is also not zero

Therefore $c_1 = 0$ and $c_2 = 0$ and so $\varphi(x) = 0$.

Case $\lambda > 0$ Do this case next time.