1.4.1. Determine the equilibrium temperature distribution for a one-dimensional rod wit constant thermal properties with the following sources and boundary conditions:

The heat equation is

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$$

Since the equilibrium temperature doesn't depend on t then $\frac{\partial u}{\partial t} = 0$. Consequently

$$K_0 \frac{d^2 u}{dx^2} = -Q \quad \text{or} \quad \frac{d^2 u}{dx^2} = -\frac{Q}{K_0}$$

(c) $Q = 0,$ $\quad \frac{\partial u}{\partial x}(0) = 0,$ $\quad u(L) = T$

Setting $Q=0$ yields $\frac{d^2u}{dx^2} = 0$, The general solution is

$$u(x) = c_1 x + c_2.$$

Solve for the boundary conditions

$$u'(x) = c_1 \quad \text{so} \quad u'(0) = 0 \quad \text{implies} \quad c_1 = 0$$

Thus $u(x) = c_2$ satisfies the first boundary condition. For the second boundary condition

$$u(L) = c_2 = T \quad \text{implies} \quad c_2 = T$$

Therefore

$$u(x) = T$$

is the equilibrium temperature.

1.4.2. Consider the equilibrium temperature distribution for a uniform one-dimensional rod with sources $Q/K_0 = x$ of thermal energy, subject to the boundary conditions $u(0) = 0$ and $u(L) = 0$.

*(a) Determine the heat energy generated per unit time inside the entire rod.

(b) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and at $x = L$.

(c) What relationships should exist between the answers in parts (a) and (b)?

The equilibrium temperature distribution is obtained by solving

$$\frac{d^2u}{dt^2} = -x \quad \text{such that } u(0)=0 \text{ and } u(L)=0$$

The general solution is $u(x) = mx + b - \frac{1}{6}x^3$. Solving for the parameters as

$$u(0) = b = 0$$

$$u(L) = mL - \frac{1}{6}L^3 = 0 \qquad m = \frac{1}{6}L^2$$

yields the equilibrium solution

$$u(x) = \frac{1}{6}L^2 x - \frac{1}{6}x^3.$$

(a) The rate of energy production inside the bar is

$$\int_0^L Q(x)\,dx = \int_0^L K_0 x\,dx = \frac{K_0}{2}x^2 \Big|_0^L = \frac{K_0}{2}L^2.$$

(b) The energy flux flowing out of the bar is

$$\phi(L) - \phi(0) = -K_0 u'(L) + K_0 u'(0).$$

Since $u'(x) = \frac{1}{6}L^2 - \frac{1}{2}x^2$ we obtain

$$\phi(L) - \phi(0) = -K_0\left(\cancel{\tfrac{1}{6}L^2} - \tfrac{1}{2}L^2 - \cancel{\tfrac{1}{6}L^2}\right) = K_0 \frac{1}{2}L^2.$$

(c) Conservation of energy states the energy produced must be equal to the energy flowing out for the bar to be in equilibrium. Thus the solution to (b) is equal the solution to (a).

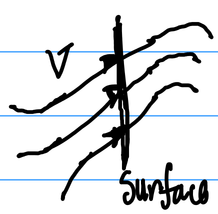
1.5.2. For conduction of thermal energy, the heat flux vector is $\boldsymbol{\phi} = -K_0 \nabla u$. If in addition the molecules move at an average velocity $\boldsymbol{V}$, a process called **convection**, then briefly explain why $\boldsymbol{\phi} = -K_0 \nabla u + c\rho u \boldsymbol{V}$. Derive the corresponding equation for heat flow, including both conduction and convection of thermal energy (assuming constant thermal properties with no sources).

The energy flux ϕ is the energy passing through a surface per unit time. Thus the units of ϕ are

$$[\phi] = \frac{[E]}{[L^2][T]}$$

where $[E]$ is dimensions of energy, $[L]$ is dimensions of length and $[T]$ stands for dimensions of time

If the molecules are moving at velocity V then the energy in those molecules will pass through any surface intersected by the direction of the velocity.



Energy passing through a surface

The energy density is $c\rho u$ where c is the heat capacity of the material and ρ its density. To obtain a flux it is enough to multiply this by the velocity, since dimensionally

$$[c\rho u] = \frac{[E]}{[L^3]} \quad \text{and} \quad [V] = \frac{[L]}{[T]}$$

implies

$$[c\rho u V] = \frac{[E]}{[L^3]}\frac{[L]}{[T]} = \frac{[E]}{[L^2][T]}$$

which is a flux.

To obtain the resulting equation for heat flow plug this flux φ into the balance of energy equation

$$c\rho \frac{\partial u}{\partial t} = -\nabla\cdot\varphi + Q$$

By vector calculus identites

$$\nabla\cdot(-K_0\nabla u + c\rho u V) = -K_0\nabla^2 u + c\rho\nabla\cdot(uV)$$

$$= -K_0\nabla^2 u + c\rho\,\nabla u\cdot V + c\rho u\nabla\cdot V$$

Consequently

$$c\rho\frac{\partial u}{\partial t} + c\rho\nabla u\cdot V + c\rho u\nabla\cdot V = K_0\nabla^2 u + Q$$

No sources means $Q=0$. Consequently setting the diffusivity

$$k = \frac{K_0}{c\rho}$$

yields

$$\frac{\partial u}{\partial t} + \nabla u\cdot V + u\nabla\cdot V = k\nabla^2 u.$$

2.3.2. Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0, \lambda = 0, \lambda < 0$). You may assume that the eigenvalues are real.

***(b)** $\phi(0) = 0$ and $\phi(1) = 0$

Case $\lambda > 0$. Then the general solution is

$$\phi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

Solving for the boundary conditions as

$$\phi(0) = A\cos(0) + B\sin(0) = A = 0$$

$$\phi(1) = B\sin(\sqrt{\lambda}) = 0$$

and noting $B \neq 0$ for ϕ to be a non-zero function, we obtain that

$\sin(\sqrt{\lambda}) = 0$ or that $\sqrt{\lambda} = n\pi$ where $n = 1, 2, \ldots$

In this case we obtain the eigenfunctions

$\phi(x) = B\sin[n\pi x]$ with $\lambda = (n\pi)^2$.

Case $\lambda = 0$. Then the general solution is

$$\phi(x) = Ax + B$$

Solving for the boundary conditions

$$\phi(0) = A0 + B = B = 0$$

and $\phi(1) = A = 0$

in which case $\phi = 0$ and so there is no eigenfunction that corresponds to $\lambda = 0$

Case $\lambda < 0$. Then the general solution is

$$\phi(x) = Ae^{\sqrt{|\lambda|}\,x} + Be^{-\sqrt{|\lambda|}\,x}$$

Again solving for the boundary conditions obtains

$$\phi(0) = Ae^{0} + Be^{-0} = A + B = 0 \qquad \text{so} \qquad B = -A$$

$$\phi(1) = Ae^{\sqrt{|\lambda|}} - Ae^{-\sqrt{|\lambda|}} = A\left(e^{\sqrt{|\lambda|}} - e^{-\sqrt{|\lambda|}}\right) = 0$$

Therefore

$$A = \frac{0}{e^{\sqrt{|\lambda|}} - e^{-\sqrt{|\lambda|}}} = 0 \qquad \text{and} \qquad B = 0$$

and there are no eigenfunctions that correspond to $\lambda < 0$

*(d) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

Case $\lambda > 0$. Then the general solution is

$$\phi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

and consequently

$$\phi'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$$

Solving for the boundary conditions as

$$\phi(0) = A\cos(0) + B\sin(0) = A = 0$$

$$\phi'(L) = B\sqrt{\lambda}\cos(\sqrt{\lambda}\,L) = 0$$

implies since $B \neq 0$ that

$\cos(\sqrt{\lambda}L) = 0$ or that $\sqrt{\lambda}L = \frac{\pi}{2} + n\pi$ for $n = 0, 1, \ldots$

In this case we obtain the eigenfunctions

$$\varphi(x) = B\sin\left(\frac{(n+\frac{1}{2})\pi x}{L}\right) \text{ with } \lambda = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2 \text{ for } n = 0, 1, \ldots$$

Case $\lambda = 0$. Then the general solution is

$$\varphi(x) = Ax + B$$

and consequently

$$\varphi'(x) = A$$

Solving for the boundary conditions

$$\varphi(0) = A0 + B = B = 0$$

and $\varphi'(L) = A = 0$

in which case $\varphi = 0$ and so there is no eigenfunction that corresponds to $\lambda = 0$

Case $\lambda < 0$. Then the general solution is

$$\varphi(x) = Ae^{\sqrt{|\lambda|}\,x} + Be^{-\sqrt{|\lambda|}\,x}$$

and consequently

$$\varphi'(x) = A\sqrt{|\lambda|}\,e^{\sqrt{|\lambda|}x} - B\sqrt{|\lambda|}\,e^{-\sqrt{|\lambda|}x}.$$

again solving for the boundary conditions obtains

$$\varphi(0) = Ae^{0} + Be^{-0} = A + B = 0 \quad \text{so} \quad B = -A$$

$$\varphi'(L) = A\sqrt{|\lambda|}\, e^{\sqrt{|\lambda|} L} + A\sqrt{|\lambda|}\, e^{-\sqrt{|\lambda|}} = A\left(e^{\sqrt{|\lambda|}} + e^{-\sqrt{|\lambda|}}\right) = 0$$

Therefore

$$A = \frac{0}{e^{\sqrt{|\lambda|}} + e^{-\sqrt{|\lambda|}}} = 0 \quad \text{and} \quad B = 0$$

and there are no eigenfunctions that correspond to $\lambda < 0$