1.4.7. For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

* **(a)** $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1, \qquad u(x,0) = f(x), \qquad \dfrac{\partial u}{\partial x}(0,t) = 1, \qquad \dfrac{\partial u}{\partial x}(L,t) = \beta$

The equilibrium solution does not depend on t so $\frac{\partial u}{\partial t} = 0$. Thus we attempt to solve the ODE

$$\frac{d^2 u}{dx^2} = -1 \text{ such that } u'(0)=1 \text{ and } u'(L)=\beta.$$

The general solution is $u(x) = mx + b - \frac{1}{2}x^2$.

Since $u'(x) = m - \frac{1}{2}x^2$ in order to satisfy the boundary conditions it must be that

$$u'(0) = m = 1 \qquad \text{so } m = 1$$

$$u'(L) = L - \tfrac{1}{2}L^2 = \beta \qquad \text{so } \beta = L - \tfrac{1}{2}L^2.$$

The only value of β for which there is an equilibrium solution is for $\beta = L - \frac{1}{2}L^2$.

Physically the energy produced within the bar and the flux leaving the bar must be the same for there to be an equilibrium solution. This uniquely determines β.

If $\beta < L - \frac{1}{2}L^2$ then more energy is being produced than leaving. That would cause the internal temperature of the bar to reach infinity, which is non-physical.

If $\beta > L - \frac{1}{2}L^2$ then more energy is leaving than being produced. Over time this would cause the temperature to go to negative infinity, which is again non-physical.

1.4.7. For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

(c) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$

Total heat at $t=0$ is

$$\int_0^L u\,dx = \int_0^L f(x)\,dx$$

The rate of change of total heat over time is

$$\frac{d}{dt}\int_0^L u\,dx = \int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + x - \beta\right)dx$$

$$\left.\frac{\partial u}{\partial x}\right|_0^L + \left.\left(\frac{x^2}{2} - \beta x\right)\right|_0^L = 0 + \frac{L^2}{2} - \beta L$$

Here the first term vanishes because of the boundary conditions

For there to be an equilibrium solution the rate of change of total heat should be zero. Thus

$$\frac{L^2}{2} - \beta L = 0$$

This implies $\beta = \frac{L}{2}$.

Only for this choice of β could there be an equilibrium solution reached as $t \to \infty$.

The equilibrium temperature does not depend on time. Thus $\partial u/\partial t = 0$ and u must satisfy

$$\frac{d^2u}{dx^2} = \beta - x \qquad \text{for} \quad x \in [0, L]$$

$$\left.\frac{du}{dx}\right|_{x=0} = 0 \qquad \left.\frac{du}{dx}\right|_{x=L} = 0$$

Integrating yields the general solution

$$\frac{du}{dx} = \int \frac{d^2u}{dx^2}\,dx = \int \left(\frac{L}{2} - x\right) dx = \frac{L}{2}x - \frac{1}{2}x^2 + C_1$$

$$u = \int \frac{du}{dx}\,dx = \int \left(\frac{L}{2}x - \frac{1}{2}x^2 + C_1\right) dx = \frac{L}{4}x^2 - \frac{1}{6}x^3 + C_1 x + C_2$$

To satisfy the boundary conditions we need

$$\left.\frac{du}{dx}\right|_{x=0} = C_1 = 0 \qquad \text{and} \qquad \left.\frac{du}{dx}\right|_{x=L} = \frac{L}{2}L - \frac{1}{2}L^2 = 0$$

To determine the value of C_2 note that the total heat at $t \to \infty$ must be the same as a $t = 0$. Thus,

$$\int_0^L \left(\frac{L}{4}x^2 - \frac{1}{6}x^3 + C_2\right) dx = \left.\frac{L}{12}x^3 - \frac{1}{24}x^4 + C_2 x \right|_0^L$$

$$= \frac{L^4}{12} - \frac{L^4}{24} + C_2 L = \frac{L^4}{24} + C_2 L = \int_0^L f(x)\,dx$$

Consequently $\quad C_2 = -\dfrac{L^3}{24} + \displaystyle\int_0^L f(x)\,dx$

2.3.3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

(a) $u(x,0) = 6\sin\frac{9\pi x}{L}$

Separation of variables as $u(x,t) = \phi(x)g(t)$ yields

$$\phi(x)g'(t) = k\,\phi''(x)g(t)$$

so that

$$\frac{g'(t)}{k\,g(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

where λ is a constant. It follows that

$$g'(t) = -\lambda k\, g(t) \qquad \text{and} \qquad \phi''(x) = -\lambda\,\phi(x)$$

The general solutions to the above ordinary differential equations is

$$g(t) = C e^{-\lambda k t} \qquad \text{and} \qquad \phi(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

The boundary conditions for ϕ are homogenous. In particular

$$\phi(0) = A\cos 0 + B\sin 0 = A = 0$$

$$\phi(L) = B\sin\sqrt{\lambda}L = 0$$

since $B \neq 0$ then $\sin\sqrt{\lambda}L = 0$ implies $\sqrt{\lambda}L = n\pi$ for $n = 1, 2, \ldots$

Consequently $\lambda = \left(\frac{n\pi}{L}\right)^2$

The superposition principle then leads to a general solution to the heat equation of the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right)$$

which satisfies the boundary conditions.

To satisfy the initial condition we need

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 6 \sin\left(\frac{9\pi x}{L}\right).$$

From the orthogonality of the sine functions we imediately infer that the only non-zero coefficient B_n occurs. when $n=9$ and that $B_9 = 6$.

Therefore

$$u(x,t) = 6 e^{-\left(\frac{9\pi}{L}\right)^2 kt} \sin\left(\frac{9\pi x}{L}\right)$$

$*$ **(c)** $u(x,0) = 2\cos\frac{3\pi x}{L}$

The general solution is again

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right)$$

but this time we need

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 2\cos\frac{3\pi x}{L}$$

By orthogonality

$$B_n = \frac{2}{L}\int_0^L 2\cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

Recall

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

so that

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$

Substituting yields

$$B_n = \frac{2}{L}\int_0^L \left(\sin\left(\frac{(3+n)\pi x}{L}\right) - \sin\left(\frac{(3-n)\pi x}{L}\right)\right)dx$$

$$= \frac{2}{L}\,\frac{-L}{(3+n)\pi}\cos\left(\frac{(3+n)\pi x}{L}\right)\Bigg|_0^L + \begin{cases} 0 & \text{if } n=3 \\ \frac{2}{L}\frac{L}{(3-n)\pi}\cos\left(\frac{(3-n)\pi x}{L}\right)\Bigg|_0^L & \text{otherwise} \end{cases}$$

$$= \frac{-2}{(3+n)\pi}\left(\cos(3+n)\pi - 1\right) + \begin{cases} 0 & \text{if } n=3 \\ \frac{2}{(3-n)\pi}\left(\cos(3-n)\pi - 1\right) & \text{otherwise} \end{cases}$$

$$= \frac{-2}{(3+n)\pi}\left((-1)^{(3+n)} - 1\right) + \begin{cases} 0 & \text{if } n=3 \\ \frac{2}{(3-n)\pi}\left((-1)^{(3-n)} - 1\right) & \text{otherwise} \end{cases}$$

Now, since $(-1)^{(3+n)} - 1 = \begin{cases} 0 & \text{when } n \text{ is odd} \\ -2 & \text{when } n \text{ is even} \end{cases}$

and $(-1)^{(3-n)} - 1 = \begin{cases} 0 & \text{when } n \text{ is odd} \\ -2 & \text{when } n \text{ is even} \end{cases}$

it follows that

$$B_n = \begin{cases} \frac{4}{(3+n)\pi} - \frac{4}{(3-n)\pi} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$$= \begin{cases} \frac{4}{\pi}\left(\frac{-2n}{9-n^2}\right) & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$$= \begin{cases} \frac{8n}{\pi(n^2-9)} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

Consequently

$$u(x,t) = \sum_{n \text{ even}} \frac{8n}{\pi(n^2-9)} e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{m=1}^{\infty} \frac{16m}{\pi(4m^2-9)} e^{-\left(\frac{2m\pi}{L}\right)^2 kt} \sin\left(\frac{2m\pi x}{L}\right)$$

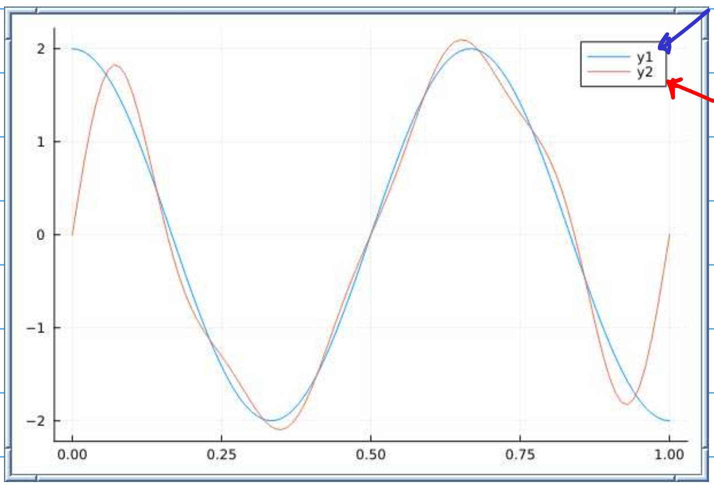
I used Julia https://julialang.org/ to numerically check the above expression using the commands

```
using Plots

b(m)=16*m/(pi*(4*m^2-9))
g(m,t)=exp(-(2*m*pi/L)^2*k*t)
phi(m,x)=sin(2*m*pi*x/L)
u(x,t)=sum([b(m)*g(m,t)*phi(m,x) for m=1:5])
u0(x)=u(x,0)

L=1
k=1
xs=0:0.01:1
plot(xs,2*cos.(3*pi*xs/L))
plot!(xs,u0.(xs))
```

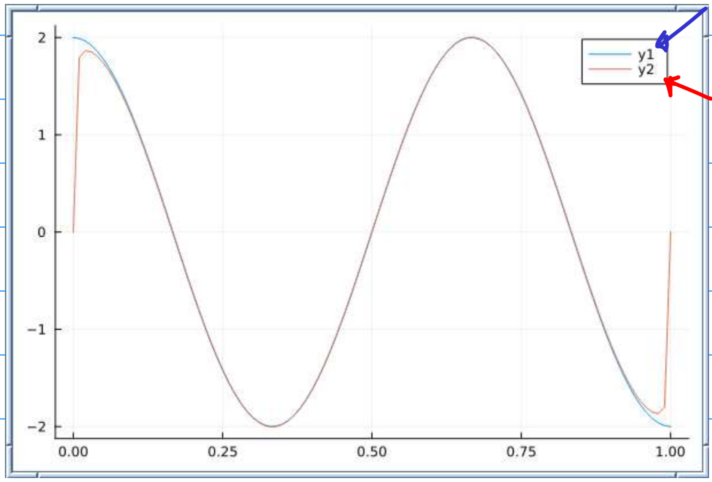
to sum the first 5 terms of the series approximation to the initial condition and compare that to $2\cos\left(\frac{3\pi x}{L}\right)$. The graph is



y_1 is desired initial cond

y_2 is the series approximation using 5 terms

Suggests the solution is correct. This can further be confirmed by summing 100 terms to obtain



y_1 is desired initial cond

y_2 is the series approximation using 100 terms

Note that, except for the discontinuity at the endpoints where the boundary condition forces the series solution to zero, that the series approximation is converging to the desired initial condition

***2.4.2.** Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0$$
$$u(L,t) = 0$$
$$u(x,0) = f(x).$$

For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

For separation of variables set $u(x,t) = \varphi(x)g(t)$ and plug this in to obtain

$$\varphi(x)g'(t) = k\varphi''(x)g(t)$$

or that

$$\frac{g'(t)}{kg(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda.$$

The homogeneous boundary conditions in x yield

$$\varphi''(x) = -\lambda\varphi(x) \quad \text{such that} \quad \varphi'(0)=0 \quad \text{and} \quad \varphi(L)=0.$$

Case $\lambda>0$, The general solution is

$$\varphi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$
$$\varphi'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

Therefore

$$\varphi'(0) = -A\sqrt{\lambda}\sin(0) + B\sqrt{\lambda}\cos(0) = B\sqrt{\lambda}$$

implies $B=0$. Now

$$\varphi(L) = A\cos\sqrt{\lambda}L = 0 \quad \text{shows} \quad \sqrt{\lambda}L = \frac{\pi}{2} + n\pi$$

Consequently,

$$\phi(x) = A\cos\left(\left(\frac{\pi}{2}+n\pi\right)\frac{x}{L}\right) \qquad \text{for } n=0,1,2,\ldots$$

Case $\lambda=0$. Then the general solution is

$$\phi(x) = Ax+B$$

and

$$\phi'(0) = A = 0 \qquad \text{followed by } \phi(L) = B = 0$$

imply there are no eigenfunctions when $\lambda=0$. Further, we don't consider $\lambda<0$ or else $g(t)$ below would grow exponentially in time.

Solving the differential equation for $g(t)$ yields

$$g(t) = Ce^{-k\lambda t} = Ce^{-k\left(\frac{\pi}{2}+n\pi\right)^2 t/L^2} \qquad \text{where } n=0,1,\ldots$$

By the superposition principle we obtain a more general solution that satisfies the boundary conditions

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-k\left(\frac{\pi}{2}+n\pi\right)^2 t/L^2} \cos\left(\left(\frac{\pi}{2}+n\pi\right)\frac{x}{L}\right)$$

where by orthogonality (assumed but could be proven) we have

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\left(\frac{\pi}{2}+n\pi\right)\frac{x}{L}\right)dx$$

Note that defining $C_n = A_{n-1}$ and shifting the index in the sums arrives at the answer in the back of the book. Namely,

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(n\pi-\frac{\pi}{2}\right)^2 t/L^2} \cos\left(\left(n\pi-\frac{\pi}{2}\right)\frac{x}{L}\right)$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\left(n\pi-\frac{\pi}{2}\right)\frac{x}{L}\right)dx.$$

2.5.1. Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions [*Hint*: Separate variables. If there are two homogeneous boundary conditions in y, let $u(x,y) = h(x)\phi(y)$, and if there are two homogeneous boundary conditions in x, let $u(x,y) = \phi(x)h(y)$.]:

*(c) $\frac{\partial u}{\partial x}(0,y) = 0$, $u(L,y) = g(y)$, $u(x,0) = 0$, $u(x,H) = 0$

The homogeneous boundaries are in the y direction thus

$$u(x,y) = h(x)\varphi(y).$$

Substituting yields

$$h''(x)\varphi(y) + h(x)\varphi''(y) = 0$$

and separating variables gives

$$-\frac{h''(x)}{h(x)} = \frac{\varphi''(y)}{\varphi(y)} = -\lambda$$

and the two ordinary differential equations

$$h''(x) = \lambda h(x) \qquad \text{and} \qquad \varphi''(y) = -\lambda\varphi(y).$$

with the boundary conditions

$$h'(0) = 0, \quad \varphi(0) = 0 \quad \text{and} \quad \varphi(H) = 0.$$

The remaining boundary condition will be solved using the superposition principle!

As in other examples the only values for λ that lead to non-zero solutions satisfying the boundary conditions are $\lambda > 0$

In that case the general solution in φ is

$$\varphi(x) = c_1 \cos\sqrt{\lambda}\,y + c_2 \sin\sqrt{\lambda}\,y$$

Further satisfying the boundary conditions yields

$$\varphi(0) = c_1 = 0$$

and

$$\varphi(H) = c_2 \sin\sqrt{\lambda}\,L = 0 \quad \text{so} \quad \sqrt{\lambda}\,H = n\pi \quad \text{for } n = 1, 2, \ldots$$

Thus

$$\varphi(y) = c_2 \sin\frac{n\pi y}{H} \qquad \text{for } n = 1, 2, \ldots$$

Solving for h such that $h''(x) = \lambda h(x)$ now yields

$$h(x) = \alpha_1 e^{\sqrt{\lambda}x} + \alpha_2 e^{-\sqrt{\lambda}x}$$

or equivalently in terms of hyperbolic functions that

$$h(x) = c_1 \cosh\sqrt{\lambda}\,x + c_2 \sinh\sqrt{\lambda}\,x.$$

Differentiating yields

$$h'(x) = c_1\sqrt{\lambda} \sinh\sqrt{\lambda}\,x + c_2\sqrt{\lambda}\cosh\sqrt{\lambda}\,x.$$

Therefore $h'(0) = c_2\sqrt{\lambda} = 0$ implies $c_2 = 0$.

Now form the superposition

$$u(x,y) = \sum_{n=1}^{\infty} b_n \cosh\frac{n\pi x}{H} \sin\frac{n\pi y}{H}$$

and solve for the remaining boundary condition.

$$u(L,y) = \sum_{n=1}^{\infty} b_n \cosh\frac{n\pi L}{H} \sin\frac{n\pi}{H}y = g(y)$$

By orthogonality of the $\sin\frac{n\pi}{H}y$ we have

$$\frac{H}{2} b_n \cosh\frac{n\pi L}{H} = \int_0^H g(y) \sin\frac{n\pi}{H}y \, dy$$

or that

$$b_n = \frac{2}{H\cosh\frac{n\pi L}{H}} \int_0^H g(y) \sin\frac{n\pi y}{H} dy.$$

Therefore, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \cosh\frac{n\pi x}{H} \sin\frac{n\pi y}{H}$$

where

$$b_n = \frac{2}{H\cosh\frac{n\pi L}{H}} \int_0^H g(y) \sin\frac{n\pi y}{H} dy.$$

***2.5.3.** Solve Laplace's equation *outside* a circular disk ($r \geq a$) subject to the boundary condition [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)\,G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

(a) $u(a,\theta) = \ln 2 + 4\cos 3\theta$

To separate variables let $u(r,\theta) = \varphi(\theta)\,G(r)$. Then

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\, \varphi(\theta)G(r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\, \varphi(\theta)G(r)}{\partial\theta^2}$$

$$= \frac{\varphi}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + \frac{G}{r^2}\frac{d^2}{d\theta^2}\varphi = 0$$

Following the hint we separate variables as

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\varphi}\frac{d^2\varphi}{d\theta^2} = \lambda$$

to obtain the ordinary differential equations

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda G \quad \text{and} \quad \varphi'' = -\lambda\varphi$$

The boundary conditions for φ are periodic in θ therefore

$$\varphi(-\pi) = \varphi(\pi) \quad \text{and} \quad \varphi'(-\pi) = \varphi'(\pi)$$

As with the heat equation in a ring we have

Case $\lambda = 0$ yields that $\varphi(\theta) = c_1$

Case $\lambda < 0$ doesnt not lead to any non-zero solutions.

Case $\lambda > 0$ yields the general solution

$$\varphi(\theta) = c_1 \cos\sqrt{\lambda}\,\theta + c_2 \sin\sqrt{\lambda}\,\theta$$

$$\varphi'(\theta) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\theta + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\theta$$

Therefore

$$\varphi(-\pi) = c_1\cos\sqrt{\lambda}\,\pi - c_2\sin\sqrt{\lambda}\,\pi$$

$$= \varphi(\pi) = c_1\cos\sqrt{\lambda}\,\pi + c_2\sin\sqrt{\lambda}\,\pi$$

and so $2c_2\sin\sqrt{\lambda}\,\pi = 0$. It follows that either

$$c_2 = 0 \quad \text{or} \quad \sqrt{\lambda}\,\pi = n\pi \quad \text{for } n = 1, 2, \ldots$$

Considering the other boundary

$$\varphi'(-\pi) = c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\pi$$

$$= \varphi'(\pi) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\pi$$

and so $2c_1\sin\sqrt{\lambda}\,\pi = 0$. It follows that either

$$c_1 = 0 \quad \text{or} \quad \sqrt{\lambda}\,\pi = n\pi \quad \text{for } n = 1, 2, \ldots$$

If $\sqrt{\lambda}\,\pi \neq n$ for any n then $c_1 = 0$ and $c_2 = 0$ in which case we have the zero solution. Thus $\sqrt{\lambda}\,\pi = n\pi$.

Therefore the eigenfunctions are

$$\varphi_0(\theta) = C_1$$

$$\varphi_n(\theta) = C_1 \cos n\theta + C_2 \sin n\theta \qquad \text{for } n = 1, 2, \ldots$$

Now solve the other ordinary differential equation

$$r \frac{d}{dr}\left(r \frac{dG_2}{dr}\right) = n^2 G_2 \qquad \text{for } n = 1, 2, \ldots$$

This is an Euler equation so substitute $G_2 = r^\alpha$ to obtain

$$r \frac{d}{dr}\left(r \frac{dr^\alpha}{dr}\right) = r \frac{d}{dr}\left(\alpha r^\alpha\right) = \alpha^2 r^\alpha = n^2 r^\alpha$$

Therefore $\alpha = \pm n$ and

$$G_2(r) = C_1 r^n + C_2 r^{-n}.$$

To avoid blow-up as $r \to \infty$ let $C_1 = 0$. Consequently

$$G_n(r) = C_2 r^{-n} \qquad \text{for } n = 1, 2, \ldots$$

When $n = 0$ we have

$$\frac{d}{dr}\left(r \frac{dG_2}{dr}\right) = 0$$

Thus

$$r \frac{dG_2}{dr} = C_1 \qquad \text{and} \qquad \frac{dG_2}{dr} = \frac{C_1}{r}$$

Then $\quad G = C_1 \log r + C_2$

Again to avoid blow-up as $r \to \infty$ set $C_1 = 0$. Thus

$$G_0(r) = C_2$$

By the superposition principle we have

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n r^{-n} \cos n\theta + B_n r^{-n} \sin n\theta \right).$$

To satisfy the boundary condition

$$u(a,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n a^{-n} \cos n\theta + B_n a^{-n} \sin n\theta \right)$$

$$= \ln 2 + 4\cos 3\theta$$

Identifying constants yields that $B_n = 0$ for all $n = 1, 2, \ldots$ and $A_0 = \ln 2$ and $A_3 = 4a^3$ with $A_n = 0$ otherwise.

The solution is

$$u(r,\theta) = \ln 2 + 4a^3 r^{-3} \cos 3\theta.$$