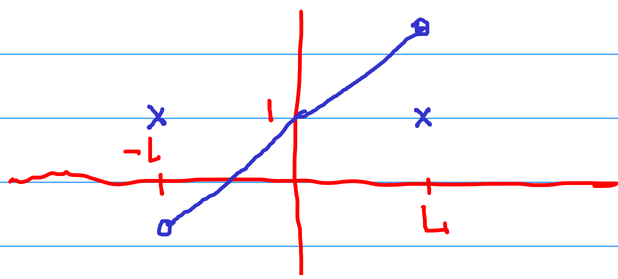
3.2.1. For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \leq x \leq L$). Compare $f(x)$ to its Fourier series:

(c) $f(x) = 1 + x$

Fourier series converges to

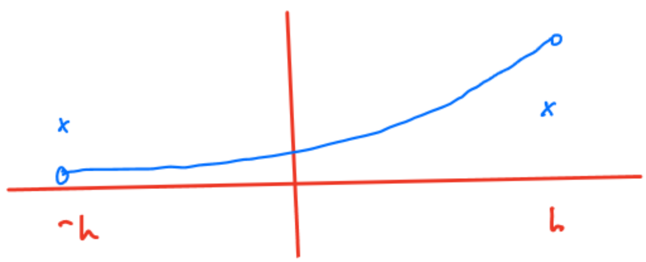


The Fourier series is different than the original function at -L and L

Since the periodic extension is not continuous at -L or L.

(d) $f(x) = e^x$

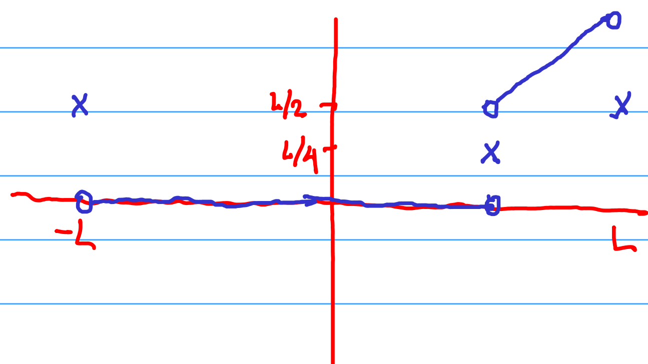
Fourier series converges to



The Fourier series is different than the original function at -L and L

Since the periodic extension is not continuous at -L or L.

(g) $f(x) = \begin{cases} x & x < L/2 \\ 0 & x > L/2 \end{cases}$

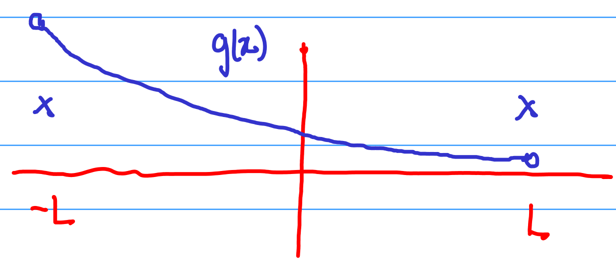


The Fourier series is different than the original function at -L, L/2 and L

Since the periodic extension is not continuous at -L or L and since the pieces in the piecewise defined function are not continuous.

3.2.2. For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$) and determine the Fourier coefficients:

(b) $f(x) = e^{-x}$



$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where $$a_0 = \frac{1}{2L}\int_{-L}^{L} e^{-x}\,dx = \frac{-1}{2L}\, e^{-x}\Big|_{-L}^{L} = \frac{1}{2L}\left(e^{L} - e^{-L}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\cos\frac{n\pi x}{L}\,dx = \frac{-1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L}\, de^{-x}$$

$$= \frac{-1}{L}\cos\frac{n\pi x}{L}\, e^{-x}\Big|_{-L}^{L} + \frac{1}{L}\int_{-L}^{L} e^{-x}\, d\cos\frac{n\pi x}{L}$$

$$= \left(\frac{1}{L}\cos n\pi\right)\left(e^{L} - e^{-L}\right) - \frac{1}{L}\int_{-L}^{L} \frac{n\pi}{L} e^{-x} \sin\frac{n\pi x}{L}\,dx$$

Now
$$\int_{-L}^{L} e^{-x}\sin\frac{n\pi x}{L}\,dx = -\int_{-L}^{L}\sin\frac{n\pi x}{L}\, de^{-x}$$

$$= -\sin\frac{n\pi x}{L}\, e^{-x}\Big|_{-L}^{L} + \int_{-L}^{L} e^{-x}\, d\sin\frac{n\pi x}{L}$$

$$= \int_{-L}^{L} \frac{n\pi}{L} e^{-x}\cos\frac{n\pi x}{L}\,dx = n\pi\, a_n$$

Consequently

$$a_n = \left(\frac{1}{L}\cos n\pi\right)\left(e^{L} - e^{-L}\right) - \frac{n^2\pi^2}{L^2} a_n$$

or that

$$a_n = \frac{\left(\frac{1}{L}\cos n\pi\right)\left(e^{L}-e^{-L}\right)}{1+n^2\pi^2/L^2} = \frac{\frac{1}{L}(-1)^n\left(e^{L}-e^{-L}\right)}{1+n^2\pi^2/L^2}$$

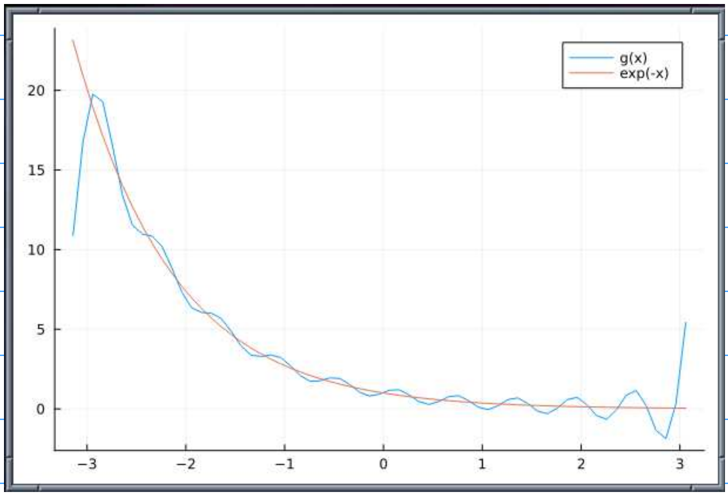
Also

$$b_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\sin\frac{n\pi x}{L}\,dx = \frac{n\pi}{L} a_n = \frac{n\pi}{L}\cdot\frac{\frac{1}{L}(-1)^n\left(e^{L}-e^{-L}\right)}{1+n^2\pi^2/L^2}$$

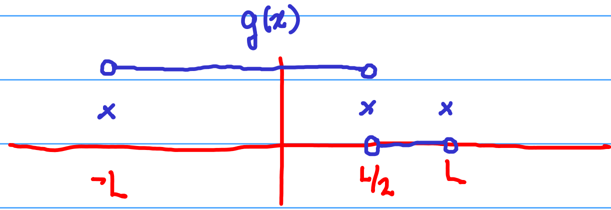
To check my answer I used Julia as

```
L=pi
a0=1/(2*L)*(exp(L)-exp(-L))
a(n)=(1/L)*(-1)^n*(exp(L)-exp(-L))/(1+n^2*pi^2/L^2)
b(n)=n*pi/L*a(n)
g(x,N)=a0+sum(a(n)*cos(n*pi*x/L)+b(n)*sin(n*pi*x/L) for n=1:N)
xs=-L:0.1:L

using Plots
plot(xs,(x->g(x,10)).(xs),label="g(x)")
plot!(xs,exp.(-xs),label="exp(-x)")
```

to sum the first 10 terms in the series and obtained



(e) $f(x) = \begin{cases} 1 & |x| < L/2 \\ 0 & |x| > L/2 \end{cases}$



$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\int_{-L}^{L/2} dx = \frac{1}{2L}\left(\frac{L}{2}+L\right) = \frac{3}{4}$

$$a_n = \frac{1}{L}\int_{-L}^{L/2} \cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L/2} = \frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{1}{n\pi}(-1)^{(n-1)/2} & \text{for } n \text{ odd} \end{cases}$$

and also

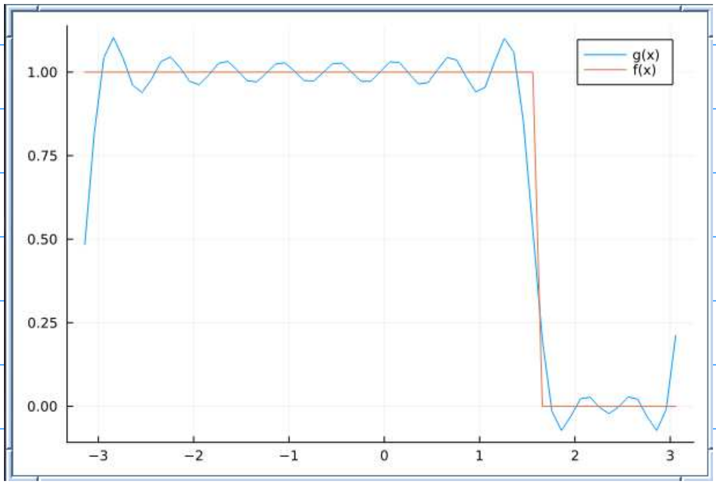
$$b_n = \frac{1}{L}\int_{-L}^{L/2} \sin\frac{n\pi x}{L}\,dx = \frac{-1}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{-L}^{L/2} = \frac{-1}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

$$= \frac{1}{n\pi}(-1)^n + \begin{cases} \frac{-1}{n\pi}(-1)^{n/2} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

To check my answer I used Julia

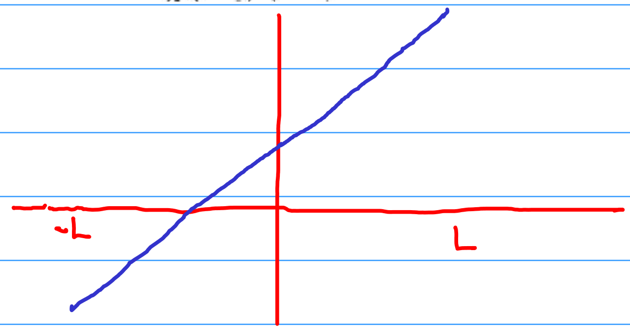
```
L=pi
a0=3/4
function a(n)
    if n%2==0
        return 0
    else
        return 1/(n*pi)*(-1)^((n-1)÷2)
    end
end
#a(n)=1/(n*pi)*sin(n*pi/2)
function b(n)
    r=1/(n*pi)*(-1)^n
    if n%2==0
        return r-1/(n*pi)*(-1)^(n÷2)
    else
        return r
    end
end
#b(n)=-1/(n*pi)*(cos(n*pi/2)-cos(n*pi))
function f(x)
    if x<L/2
        return 1
    else
        return 0
    end
end
g(x,N)=a0+sum(a(n)*cos(n*pi*x/L)+b(n)*sin(n*pi*x/L) for n=1:N)
xs=-L:0.1:L

using Plots
plot(xs,(x->g(x,10)).(xs),label="g(x)")
plot!(xs,f.(xs),label="f(x)")
```

to sum the first 10 terms and obtained

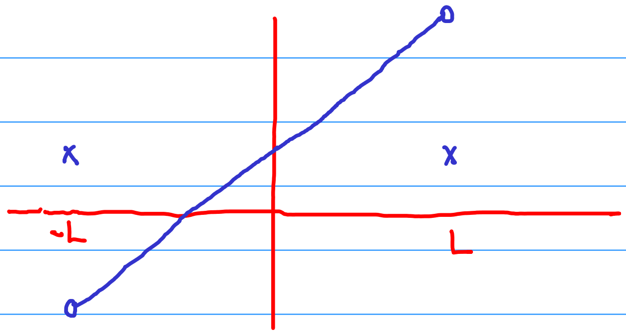


3.3.1. For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$:

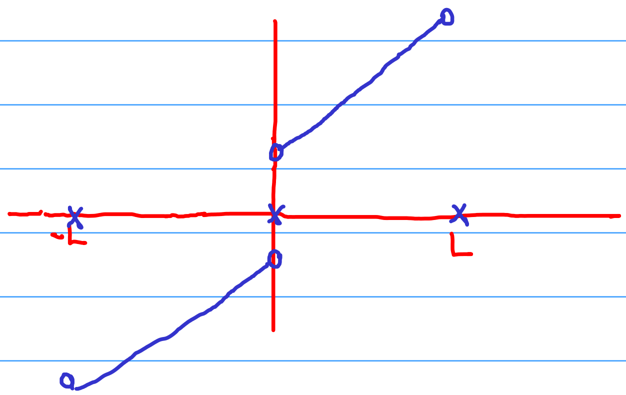
(b) $f(x) = 1 + x$



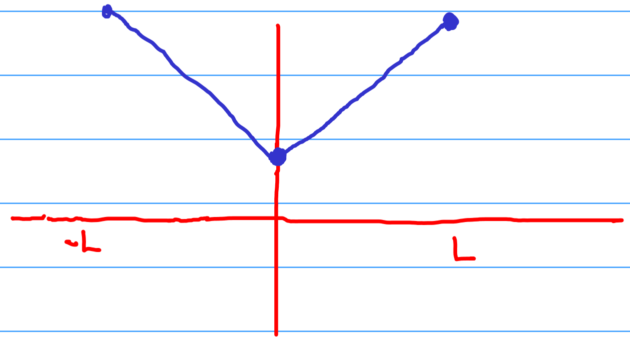
The Fourier series of f(x) is



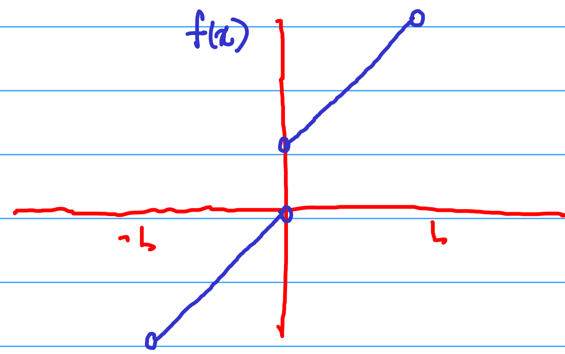
The Fourier sine series of f(x) is



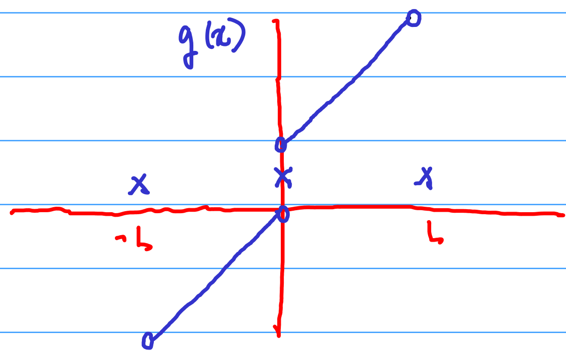
The Fourier cosine series of f(x) is



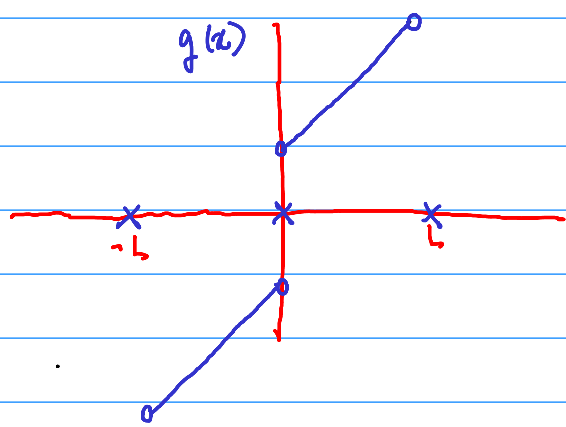
(c) $f(x) = \begin{cases} x & x < 0 \\ 1 + x & x > 0 \end{cases}$



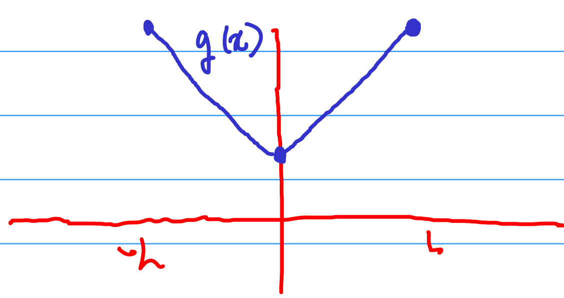
The Fourier series of f(x) is



The Fourier sine series of f(x) is



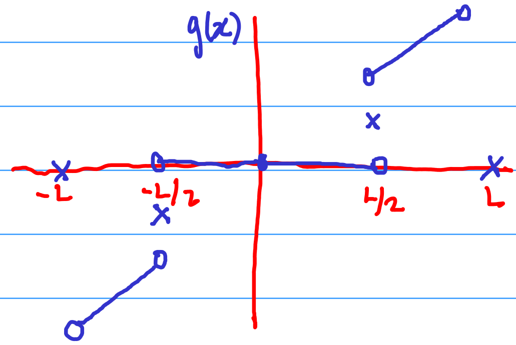
The Fourier cosine series of f(x) is



3.3.2. For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients:

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

The Fourier sine series is



$$g(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx = \frac{2}{L}\int_{L/2}^{L} x \sin\frac{n\pi x}{L}\, dx = \frac{2}{L}\int_{L/2}^{L} \frac{-L}{n\pi}\, x\, d\cos\frac{n\pi x}{L}$$

$$= \frac{-2}{n\pi}\, x \cos\frac{n\pi x}{L}\Big|_{L/2}^{L} + \frac{2}{n\pi}\int_{L/2}^{L} \cos\frac{n\pi x}{L}\, dx$$

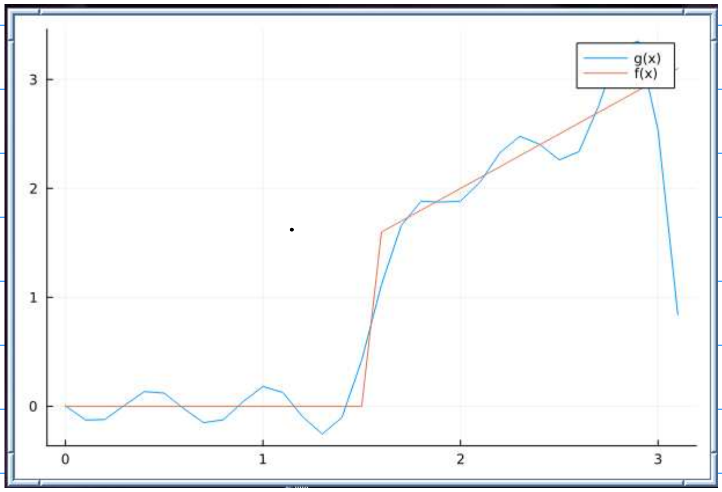
$$= \frac{-2}{n\pi}\left(L\cos n\pi - \frac{L}{2}\cos\frac{n\pi}{2}\right) + \frac{2}{n\pi}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{L/2}^{L}$$

$$= \frac{-2L}{n\pi}(-1)^n + \frac{L}{n\pi}\begin{cases} (-1)^{n/2} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} - \frac{2L}{n^2\pi^2}\begin{cases} 0 & \text{for } n \text{ even} \\ (-1)^{(n-1)/2} & \text{for } n \text{ odd} \end{cases}$$

To check my answer I summed 10 terms of the series

```
L=pi
function b(n)
    r=-2*L/(n*pi)*(-1)^n
    if n%2==0
        return r+L/(n*pi)*(-1)^(n÷2)
    else
        return r-2*L/(n^2*pi^2)*(-1)^((n-1)÷2)
    end
end
function f(x)
    if x<L/2
        return 0
    else
        return x
    end
end
g(x,N)=sum(b(n)*sin(n*pi*x/L) for n=1:N)
xs=0:0.1:L

using Plots
plot(xs,(x->g(x,10)).(xs),label="g(x)")
plot!(xs,f.(xs),label="f(x)")
```

with Julia led to



3.3.7. Show that e^x is the sum of an even and an odd function.

$$e^x = \frac{1}{2}e^x - \frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} + \frac{1}{2}e^x$$

$$= \underbrace{\frac{1}{2}(e^x - e^{-x})}_{\text{odd}} + \underbrace{\frac{1}{2}(e^{-x} + e^x)}_{\text{even}} = f_o(x) + f_e(x)$$

where f_o is the odd function

$$f_o(x) = \frac{1}{2}(e^x - e^{-x})$$

and f_e is the even function

$$f_e(x) = \frac{1}{2}(e^x + e^{-x})$$

3.4.1. The integration-by-parts formula

$$\int_a^b u \frac{dv}{dx}\, dx = uv\Big|_a^b - \int_a^b v \frac{du}{dx}\, dx$$

is known to be valid for functions $u(x)$ and $v(x)$, which are continuous and have continuous first derivatives. However, we will assume that u, v, du/dx, and dv/dx are continuous only for $a \le x \le c$ and $c \le x \le b$; we assume that all quantities may have a jump discontinuity at $x = c$.

*(a) Derive an expression for $\int_a^b u\, dv/dx\, dx$ in terms of $\int_a^b v\, du/dx\, dx$.

Since

$$\int_a^b u \frac{dv}{dx}\, dx = \int_a^c u \frac{dv}{dx} dx + \int_c^b u \frac{dv}{dx} dx$$

and $u, v, \frac{du}{dv}$ and $\frac{dv}{du}$ are continuous on these intervals then

$$\int_a^c u \frac{dv}{dx} dx = uv\Big|_a^{c^-} - \int_a^c v \frac{du}{dx}\, dx$$

$$\int_c^b u \frac{dv}{dx} dx = uv\Big|_{c^+}^{b} - \int_c^b v \frac{du}{dx}\, dx$$

where c^- and c^+ indicate limits from the left and right respectively.

Putting these together yields

$$\int_a^b u \frac{dv}{dx}\, dx = uv\Big|_a^{c^-} + uv\Big|_{c^+}^{b} - \int_a^b v \frac{du}{dx}\, dx$$

$$= uv\Big|_a^{b} + uv\Big|_{c^+}^{c^-} - \int_a^b v \frac{du}{dx}\, dx$$

(b) Show that this reduces to the integration-by-parts formula if u and v are continuous across $x = c$. It is *not* necessary for du/dx and dv/dx to be continuous at $x = c$.

In the case that u and v are continuous then

$$u(c^+) = u(c^-) \quad \text{and} \quad v(c^+) = v(c^-)$$

Therefore

$$uv\Big|_{c^+}^{c^-} = u(c^-)v(c^-) - u(c^+)v(c^+) = 0$$

and the formula becomes

$$\int_a^b u \frac{dv}{dx}\, dx = uv\Big|_a^b - \int_a^b v \frac{du}{dx}\, dx$$

which is the usual integration by parts formula.

3.5.1. Consider

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{3.5.12}$$

(a) Determine b_n from (3.3.11), (3.3.12), and (3.5.6).

(b) For what values of x is (3.5.12) an equality?

*(c) Derive the Fourier cosine series for x^3 from (3.5.12). For what values of x will this be an equality?

(a) Recall

$$x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \qquad -L < x < L. \tag{3.3.11}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx = \frac{2}{L}\int_0^L x \sin \frac{n\pi x}{L}\, dx = \frac{2L}{n\pi}(-1)^{n+1}. \tag{3.3.12}$$

Also

$$\frac{x^2}{2} = \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi x}{L} + \frac{\sin 3\pi x/L}{3^3} + \frac{\sin 5\pi x/L}{5^3} + \cdots\right). \tag{3.5.6}$$

Substituting (3.3.1) into (3.56) yields

$$\frac{x^2}{2} = \frac{L}{2}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi x}{L} + \frac{1}{3^3}\sin\frac{3\pi x}{L} + \frac{1}{5^3}\sin\frac{5\pi x}{L} + \cdots\right)$$

$$= \frac{L}{2}\sum_{n=1}^{\infty}\frac{2L(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{L} - \frac{4L^2}{\pi^3}\sum_{n \text{ odd}}\frac{1}{n^3}\sin\frac{n\pi x}{L}$$

$$= \frac{L^2}{\pi}\sum_{n \text{ odd}}\frac{1}{n}\sin\frac{n\pi x}{L} - \frac{L^2}{\pi}\sum_{n \text{ even}}\frac{1}{n}\sin\frac{n\pi x}{L} - \frac{4L^2}{\pi^3}\sum_{n \text{ odd}}\frac{1}{n^3}\sin\frac{n\pi x}{L}$$

$$= \frac{L^2}{\pi}\sum_{n \text{ odd}}\left(\frac{1}{n} - \frac{4}{\pi^2 n^3}\right)\sin\frac{n\pi x}{L} - \frac{L^2}{\pi}\sum_{n \text{ even}}\frac{1}{n}\sin\frac{n\pi x}{L}$$

Therefore

$$b_n = \begin{cases} 2\frac{L^2}{\pi}\left(\frac{1}{n} - \frac{4}{\pi^2 n^3}\right) & \text{for } n \text{ odd} \\ -2\frac{L^2}{\pi}\frac{1}{n} & \text{for } n \text{ even} \end{cases}$$

(b) Since the odd extension of x^2 is smooth everywhere except $x=L$ then (3.5.12) is an equality on $[0,L)$, at $x=L$ it will converge to the average $\frac{0+L^2}{2} = \frac{L^2}{2}$.

(c) The Fourier cosine series may be obtained by integrating term by term. Thus,

$$\frac{x^3}{3} \sim \sum_{n=1}^{\infty} \int_0^x b_n \sin\frac{n\pi s}{L}\, ds = \sum_{n=1}^{\infty} \frac{-b_n L}{n\pi} \cos\frac{n\pi s}{L} \Big|_0^x$$

$$= \sum_{n=1}^{\infty} \frac{-b_n L}{n\pi}\left(\cos\frac{n\pi x}{L} - 1\right) = \sum_{n=1}^{\infty} \frac{b_n L}{n\pi} + \sum_{n=1}^{\infty} \frac{-b_n L}{n\pi} \cos\frac{n\pi x}{L}$$

Therefore

$$x^3 = \sum_{n=0}^{\infty} a_0 \cos\frac{n\pi x}{L}$$

where

$$a_0 = 3\sum_{n=1}^{\infty} \frac{b_n L}{n\pi} \quad \text{and} \quad a_n = \frac{-3 b_n L}{n\pi} \quad \text{for } n > 0$$

Here again

$$b_n = \begin{cases} 2\frac{L^2}{\pi}\left(\frac{1}{n} - \frac{4}{\pi^2 n^3}\right) & \text{for } n \text{ odd} \\ -2\frac{L^2}{\pi}\frac{1}{n} & \text{for } n \text{ even}. \end{cases}$$

simplifying, obtains

$$a_0 = 6\frac{L^3}{\pi^2}\left(\sum_{n \text{ odd}} \left(\frac{1}{n^2} - \frac{4}{\pi^2 n^4}\right) - \sum_{n \text{ even}} \frac{1}{n^2}\right)$$

$$= 6\frac{L^3}{\pi^2}\left(\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^{n+1} - \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}\right)$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^{n+1} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots = \frac{\pi^2}{12}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$$

as seen by Maple (Wolfram Alpha also given same)

```
> sum(1/n^2*(-1)^(n+1),n=1..infinity);
                                         2
                                       Pi
                                       ---
                                       12

> sum(1/(2*k+1)^4,k=0..infinity);
                                         4
                                       Pi
                                       ---
                                       96
```

see below for an easier way to find a_0

Therefore

$$a_0 = \frac{6L^3}{\pi^2}\left(\frac{\pi^2}{12} - \frac{4}{\pi^2}\frac{\pi^4}{96}\right) = 6L^3\left(\frac{1}{12} - \frac{1}{24}\right) = \frac{L^3}{4}$$

Note, a simpler way to find a_0 is from the definition

$$a_0 = \frac{1}{L}\int_0^L x^3\,dx = \frac{1}{4L}x^4\Big|_0^L = \frac{L^3}{4}.$$

It follows that

$$x^3 = \sum_{n=0}^{\infty} a_0 \cos\frac{n\pi x}{L}$$

where

$$a_0 = \frac{L^3}{4}$$

and for $n>0$ that

$$a_n = \begin{cases} 6\frac{L^3}{\pi^2}\left(\frac{1}{n^2} - \frac{4}{\pi^2 n^4}\right) & \text{for } n \text{ odd} \\ -6\frac{L^3}{\pi^2}\frac{1}{n^2} & \text{for } n \text{ even}. \end{cases}$$