HW4 problems 4.2.2, 4.4.1, 4.4.2, 4.4.5, 4.4.10, 5.3.1 5.3.3

4.2.2. Show that c^2 has the dimensions of velocity squared.

The wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The dimensions of $\frac{\partial^2 u}{\partial t^2}$ are

$$\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{[u]}{[T]^2}.$$

The dimensions of $\frac{\partial^2 u}{\partial x^2}$ are

$$\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{[u]}{[L]^2}.$$

Therefore, the dimensions of c^2 are

$$[c^2] = \frac{\left[\frac{\partial^2 u}{\partial t^2}\right]}{\left[\frac{\partial^2 u}{\partial x^2}\right]} = \frac{\frac{[u]}{[T]^2}}{\frac{[u]}{[L]^2}} = \frac{[L]^2}{[T]^2}.$$

It follows that the dimensions of c are

$$[c] = \frac{[L]}{[T]}$$

which is a velocity.

4.4.1. Consider vibrating strings of uniform density ρ_0 and tension T_0.

*(a) What are the natural frequencies of a vibrating string of length L fixed at both ends?

*(b) What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u/\partial x(H, t) = 0$]? Sketch a few modes of vibration as in Fig. 4.4.1.

(c) Show that the modes of vibration for the *odd* harmonics (i.e., $n = 1, 3, 5, \ldots$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

(a) Consider $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ where $c^2 = \dfrac{T_0}{\rho_0}$.

If the boundary conditions are fixed as

$$u(0,t) = a \quad \text{and} \quad u(L,t) = b$$

then consider

$$w(x,t) = u(x,t) - a - \frac{(b-a)}{L} x.$$

Since $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 u}{\partial t^2}$ and $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial^2 u}{\partial x^2}$ it follows that w satisfies

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

with the homogenous boundary conditions

$$w(0,t) = 0 \quad \text{and} \quad w(L,t) = 0$$

Performing separation of variables

$$w(x,t) = g(x)\, h(t)$$

yields

$$\varphi(x)h''(t) = c^2\varphi''(x)h(t)$$

so that

$$\frac{h''(t)}{c^2 h(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda .$$

Therefore

$$h''(t) = -\lambda c^2 h(t) \quad \text{and} \quad \varphi''(x) = -\lambda\varphi(x).$$

In general

$$\varphi(x) = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x$$

while the boundary conditions

$$\varphi(0)=0 \quad \text{and} \quad \varphi(L)=0$$

imply

$$\varphi(0) = a\cos\sqrt{\lambda}\,0 + b\sin\sqrt{\lambda}\,0 = a = 0$$

and $\varphi(L) = b\sin\sqrt{\lambda}L = 0.$

since $b \neq 0$ then $\sin\sqrt{\lambda}L = 0$ or $\sqrt{\lambda}L = n\pi.$

Now

$$h(t) = \alpha\cos(c\sqrt{\lambda}t) + \beta\sin(c\sqrt{\lambda}t)$$
$$= \alpha\cos\frac{cn\pi t}{L} + \beta\sin\frac{cn\pi t}{L}.$$

By supperposition

$$w(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{cn\pi t}{L} + b_n \sin \frac{cn\pi t}{L} \right) \cos \frac{n\pi x}{L}$$

The natural frequencies can be seen in the time dependent part of the above expression. Namely, the frequencies are

$$\frac{cn\pi}{L} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

(b) If one end of the string is free, then we have the boundary conditions

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi'(H) = 0$$

again taking

$$\varphi(x) = a \cos \sqrt{\lambda}\, x + b \sin \sqrt{\lambda}\, x$$

yields

$$\varphi(0) = a = 0$$

$$\varphi'(x) = b\sqrt{\lambda} \cos \sqrt{\lambda}\, x$$

$$\varphi'(H) = b\sqrt{\lambda} \cos \sqrt{\lambda}\, H = 0$$

since $b \neq 0$ then $\sqrt{\lambda}\, H = (n + \frac{1}{2})\pi$ for $= 0, 1, 2, \ldots$

By superposition

$$w(x,t) = \sum_{n=0}^{\infty}\left(a_n \cos \frac{c(n+\frac{1}{2})\pi t}{H} + b_n \sin \frac{c(n+\frac{1}{2})\pi t}{H}\right) \sin \frac{(n+\frac{1}{2})\pi x}{H}$$

In this case the natural frequencies are

$$\frac{c(n+\frac{1}{2})\pi}{H} \quad \text{for } n = 0, 1, 2, \ldots.$$

(c) Taking $H = L/2$ in part (b) yields the modes

$$\sin \frac{(n+\frac{1}{2})\pi x}{H} = \sin \frac{(n+\frac{1}{2})\pi x}{L/2} = \sin \frac{(2n+1)\pi x}{L}.$$

for $n = 0, 1, 2, \ldots$,

Taking $n = 2k+1$ in part (a) yields the modes

$$\sin \frac{n\pi x}{L} = \sin \frac{(2k+1)\pi x}{L} \quad \text{for } k = 0, 1, 2, \ldots.$$

These are both the same collection of modes.

4.4.2. In Section 4.2 it was shown that the displacement u of a nonuniform string satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + Q,$$

where Q represents the vertical component of the body force per unit length. If $Q = 0$, the partial differential equation is homogeneous. A slightly different homogeneous equation occurs if $Q = \alpha u$.

(a) Show that if $\alpha < 0$, the body force is restoring (toward $u = 0$). Show that if $\alpha > 0$, the body force tends to push the string further away from its unperturbed position $u = 0$.

(b) Separate variables if $\rho_0(x)$ and $\alpha(x)$ but T_0 is constant for physical reasons. Analyze the time-dependent ordinary differential equation.

***(c)** Specialize part (b) to the constant coefficient case. Solve the initial value problem if $\alpha < 0$:

$$u(0,t) = 0, \qquad u(x,0) = 0$$

$$u(L,t) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = f(x).$$

What are the frequencies of vibration?

(a) Suppose $\alpha < 0$ then $\alpha = -|\alpha|$ and

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - |\alpha| u$$

Since the force term $-|\alpha| u$ is negative when $u > 0$ that tends to push u down. On the other hand $u < 0$ leads to a positive force which tends to push u upwards.

Suppose $\alpha > 0$ then $\alpha = |\alpha|$ and

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + |\alpha| u$$

In this case $u > 0$ results in a positive force that pushes u further up and $u < 0$ results in a negative force that will push u further down.

(b) We have

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha(x)\, u$$

For separation of variables let

$$u(x,t) = \varphi(x) h(t).$$

Then

$$\rho_0(x)\varphi(x) h''(t) = T_0 \varphi''(x) h(t) + \alpha(x)\varphi(x) h(t)$$

Dividing by $h(t)$ yields

$$\rho_0(x)\varphi(x)\frac{h''(t)}{h(t)} = T_0\varphi''(x) + \alpha(x)\varphi(x)$$

and then

$$\frac{h''(t)}{h(t)} = \frac{T_0\varphi''(x) + \alpha(x)\varphi(x)}{\rho_0(x)\varphi(x)} = -\lambda$$

Consequently

$$h''(t) = -\lambda h(t) \quad \text{and} \quad T_0\varphi''(x) = \left(-\lambda \rho_0(x) - \alpha(x)\right)\varphi(x)$$

Analysis of the the time dependent consists of three cases.

Case $\lambda < 0$ then $-\lambda = |\lambda|$ and

$$h(t) = a e^{-\sqrt{|\lambda|}\,t} + b e^{\sqrt{|\lambda|}\,t}$$

or equivalently

$$h(t) = a\cosh\sqrt{|\lambda|}\,t + b\sinh\sqrt{|\lambda|}\,t$$

In this case the time dependent behavior is exponential.

Case $\lambda = 0$ then

$$h(t) = at + b$$

and the time behavior is linear.

Case $\lambda > 0$ then $-\lambda = -|\lambda|$

$$h(t) = a\cos\sqrt{|\lambda|}\,t + b\sin\sqrt{|\lambda|}\,t$$

and the time behavior is oscillatory.

(c) The constant coefficient case with $\alpha < 0$ is

$$\frac{h''(t)}{h(t)} = \frac{T_0\varphi''(x) + \alpha\varphi(x)}{\rho_0\varphi(x)} = -\lambda$$

So that

$$\varphi''(x) = \frac{|\alpha| - \lambda\rho_0}{T_0}\varphi(x)$$

subject to the boundary conditions

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(L) = 0.$$

If $\frac{|\alpha| - \lambda \rho_0}{T_0} \geq 0$ it is known that there are no non-zero solutions which satisfy the boundary conditions. Thus, we assume $\lambda \rho_0 > |\alpha|$.

In this case the general solution is

$$\varphi(x) = a \cos\left(\sqrt{\frac{\lambda \rho_0 - |\alpha|}{T_0}}\, x\right) + b \sin\left(\sqrt{\frac{\lambda \rho_0 - |\alpha|}{T_0}}\, x\right)$$

The boundary conditions then imply

$$\varphi(0) = a = 0$$
$$\varphi(L) = b \sin\left(\sqrt{\frac{\lambda \rho_0 - |\alpha|}{T_0}}\, L\right) = 0$$

and so $\sqrt{\frac{\lambda \rho_0 - |\alpha|}{T_0}}\, L = n\pi$ for $n = 1, 2, 3, \ldots$

It follows that

$$\frac{\lambda \rho_0 - |\alpha|}{T_0} = \left(\frac{n\pi}{L}\right)^2, \qquad \lambda \rho_0 = |\alpha| + T_0 \left(\frac{n\pi}{L}\right)^2$$

and finally $\lambda = \frac{|\alpha| + T_0 \left(\frac{n\pi}{L}\right)^2}{\rho_0} > 0$

for $n = 1, 2, 3, \ldots$

Therefore, the time dependent part is oscillatory and by the superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\sqrt{\frac{|\alpha| + T_0 \left(\frac{n\pi}{L}\right)^2}{\rho_0}}\, t \right) + b_n \sin\left(\sqrt{\frac{|\alpha| + T_0 \left(\frac{n\pi}{L}\right)^2}{\rho_0}}\, t \right) \right] \sin\frac{n\pi x}{L}$$

Now solve for a_n and b_n using the initial conditions.

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = 0$$

implies $a_n = 0$ for all $n = 1, 2, \ldots$ Thus

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\sqrt{\frac{|\alpha| + T_0 \left(\frac{n\pi}{L}\right)^2}{\rho_0}}\, t \right) \sin\frac{n\pi x}{L}$$

and differentiating yields

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \sqrt{\frac{|\alpha| + T_0 \left(\frac{n\pi}{L}\right)^2}{\rho_0}} \sin\frac{n\pi x}{L} = f(x).$$

It follows that

$$b_n \sqrt{\frac{|\alpha| + T_0 \left(\frac{n\pi}{L}\right)^2}{\rho_0}} = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

or equivalently

$$b_n = \sqrt{\frac{\rho_0}{|\alpha| + T_0 \left(\frac{n\pi}{L}\right)^2}}\, \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx .$$

4.4.5. Redo Exercise 4.4.3(b) if $4\pi^2\rho_0 T_0/L^2 < \beta^2 < 16\pi^2\rho_0 T_0/L^2$.

Recall 4.4.3 is

4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

***(b)** Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0/L^2$).

(a) The total energy in the string is given by

$$E = \frac{1}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \qquad \text{where } c^2 = \frac{T_0}{\rho_0}$$

Therefore

$$\frac{\partial E}{\partial t} = \frac{1}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$= \int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right) dx$$

$$= \int_0^L \left[\frac{\partial u}{\partial t}\left(c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\beta}{T_0}\frac{\partial u}{\partial t}\right) + c^2 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right] dx$$

Now since

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) = \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}$$

then

$$\frac{\partial E}{\partial t} = \int_0^L \left[c^2 \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) - \cancel{\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}} \right) - \frac{\beta}{T_0}\left(\frac{\partial u}{\partial t}\right)^2 + \cancel{c^2 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}} \right] dx$$

$$= c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L - \frac{\beta}{T_0}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx$$

Assuming the boundary terms vanish as they do for part (b) then we have

$$\frac{\partial E}{\partial t} = -\frac{\beta}{T_0}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dt$$

This shows that if $\beta > 0$ then $\frac{\partial E}{\partial t} \leq 0$ and so the energy is decreasing over time. This is damping.

On the other hand, if $\beta < 0$ then $\frac{\partial E}{\partial t} > 0$ and so the energy is increasing over time, which is not damping.

If $\beta = 0$ the term vanishes and the system conserves energy.

A briefer explanation appears in Lecture 15 in the notes.

(b) For separation of variables

$$u(x,t) = \varphi(x)\, h(t)$$

Then

$$\rho_0 \varphi(x)\, h''(t) = T_0 \varphi''(x) h(t) - \beta h'(t)$$

$$\frac{\rho_0}{T_0} \frac{h''(t)}{h(t)} = \frac{\varphi''(x)}{\varphi(x)} - \frac{\beta}{T_0} \frac{h'(t)}{h(t)}.$$

Thus,

$$\frac{\rho_0}{T_0} \frac{h''(t)}{h(t)} + \frac{\beta}{T_0} \frac{h'(t)}{h(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

we obtain

$$\rho_0 h''(t) + \beta h'(t) = -\lambda T_0 h(t) \quad \text{and} \quad \varphi''(x) = -\lambda \varphi(x)$$

The general solution for $\varphi(x)$ is

$$\varphi(x) = A \cos\sqrt{\lambda}x + B \sin\sqrt{\lambda}x$$

The boundary conditions $\varphi(0) = 0$ and $\varphi(L) = 0$ then imply

$$\varphi(0) = A = 0$$

$$\varphi(L) = B \sin\sqrt{\lambda}L = 0 \quad \text{so} \quad \sqrt{\lambda}L = n\pi$$

$$\text{or} \quad \sqrt{\lambda} = \frac{n\pi}{L} \quad \text{for } n = 1, 2, \ldots$$

Now solve for $h(t)$ by plugging in e^{rt}

$$\rho_0 r^2 e^{rt} + \beta r e^{rt} = -\lambda T_0 e^{rt}$$

implies $\rho_0 r^2 + \beta r + \lambda T_0 = 0$.

By the quadratic formula with $a = \rho_0$, $b = \beta$ and $c = \lambda T_0$ we obtain that

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \lambda T_0}}{2\rho_0} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0}}{2\rho_0}$$

The condition $4\pi^2 \rho_0 T_0 / L^2 < \beta^2 < 16\pi^2 \rho_0 T_0 / L^2$ implies

$$\beta^2 - 4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 > 0 \quad \text{for } n = 1$$
$$\beta^2 - 4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 < 0 \quad \text{for } n = 2, 3, 4, \ldots$$

Therefore, break the solution into two cases.

Case $n=1$

$$h(t) = a_1 \cosh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0 \left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right) + b_1 \sinh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0 \left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right)$$

Case $n=2$

$$h(t) = e^{-\frac{\beta}{2\rho_0} t} \left[a_n \cos\left(\frac{\sqrt{4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) + b_n \sin\left(\frac{\sqrt{4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) \right]$$

By superposition it follows that

$u(x,t) =$

$$\left[a_1 \cosh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right) + b_1 \sinh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right)\right] \sin\frac{\pi x}{L}$$

$$+ \sum_{n=2}^{\infty} e^{\frac{-\beta}{2\rho_0}t} \left[a_n \cos\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) + b_n \sin\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right)\right] \sin\frac{n\pi x}{L}.$$

Solving for the initial conditions yields

$$u(x,0) = \sum a_n \sin\frac{n\pi x}{L} = f(x)$$

and so $\quad a_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$

Then

$\frac{\partial u}{\partial t}(x,t) =$

$$\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0}\right)\left[a_1 \sinh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right) + b_1 \cosh\left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0} t\right)\right] \sin\frac{\pi x}{L}$$

$$+ \sum_{n=2}^{\infty} \left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0}\right) e^{\frac{-\beta}{2\rho_0}t} \left[-a_n \sin\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) + b_n \cos\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right)\right] \sin\frac{n\pi x}{L}$$

$$+ \sum_{n=2}^{\infty} \frac{-\beta}{2\rho_0} e^{\frac{-\beta}{2\rho_0}t} \left[a_n \cos\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right) + b_n \sin\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0} t\right)\right] \sin\frac{n\pi x}{L}$$

Consequently

$$\frac{\partial u}{\partial t}(x,0) = \left(\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}}{2\rho_0}\right) b_1 \sin\frac{\pi x}{L} + \sum_{n=2}^{\infty}\left[\left(\frac{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}{2\rho_0}\right) b_n - \frac{\beta}{2\rho_0} a_n\right] \sin\frac{n\pi x}{L}$$

and it follows that

$$b_1 = \frac{2\rho_0}{-\beta + \sqrt{\beta^2 - 4\rho_0\left(\frac{\pi}{L}\right)^2 T_0}} \, \frac{2}{L}\int_0^L g(x)\sin\frac{\pi x}{L}\,dx$$

and for $n \geq 2$ that

$$b_n = \frac{2\rho_0}{\sqrt{4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0 - \beta^2}}\left(\frac{\beta}{2\rho_0} a_n + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right).$$

4.4.10. What happens to the total energy E of a vibrating string (see Exercise 4.4.9)

(a) If $u(0,t)=0$ and $u(L,t)=0$?

(b) If $\frac{\partial u}{\partial x}(0,t)=0$ and $u(L,t)=0$?

(c) If $u(0,t)=0$ and $\frac{\partial u}{\partial x}(L,t)=-\gamma u(L,t)$ with $\gamma>0$?

(d) If $\gamma<0$ in part (c)?

Recall from Exercise 4.4.9 that

$$\frac{dE}{dt}=c^2\frac{\partial u}{\partial x}.\frac{\partial u}{\partial t}\bigg|_0^L$$

(a) Since $\frac{\partial u}{\partial t}(0,t)=\frac{\partial 0}{\partial t}=0$ and $\frac{\partial u}{\partial t}(L,t)=\frac{\partial 0}{\partial t}=0$ then

$$\frac{dE}{dt}=c^2\left[\frac{\partial u}{\partial x}(L,t)\cancel{\frac{\partial u}{\partial t}(L,t)}-\frac{\partial u}{\partial x}(0,t)\cancel{\frac{\partial u}{\partial t}(0,t)}\right]=0.$$

Consequently the energy is conserved.

(b) In this case

$$\frac{dE}{dt}=c^2\left[\frac{\partial u}{\partial x}(L,t)\cancel{\frac{\partial u}{\partial t}(L,t)}-\cancel{\frac{\partial u}{\partial x}(0,t)}\frac{\partial u}{\partial t}(0,t)\right]=0.$$

Consequently the energy is conserved.

(c) $$\frac{dE}{dt}=c^2\left[\frac{\partial u}{\partial x}(L,t)\frac{\partial u}{\partial t}(L,t)-\frac{\partial u}{\partial x}(0,t)\cancel{\frac{\partial u}{\partial t}(0,t)}\right]$$

$$=c^2\left[-\gamma u(L,t)\frac{\partial u}{\partial t}(L,t)\right]=\frac{-\gamma c^2}{2}\frac{d}{dt}u^2(L,t)$$

Integration then yields that

$$E(t) - E(0) = \frac{-\gamma c^2}{2}\left[u^2(L,t) - u^2(L,0)\right]$$

or

$$E(t) = E(0) - \frac{\gamma c^2}{2}\left[u^2(L,t) - u^2(L,0)\right].$$

Since $\gamma > 0$ then $u^2(L,t) > u^2(L,0)$ implies this term decreases the total energy while $u^2(L,t) < u^2(L,0)$ leads to an increase. I think the effect will stabalize the amount of energy in the system but am not sure.

(d) When $\gamma < 0$ then $u^2(L,t) > u^2(L,0)$ implies this term increases the total energy. This may cause total energy to increase.

***5.3.1.** Do Exercise 4.4.2(b). Show that the partial differential equation may be put into Sturm–Liouville form.

The equation in Exercise 4.4.2 part (b) is

$$T_0 \phi''(x) = \left(-\lambda \rho_0(x) - \alpha(x)\right) \phi(x)$$

We now seek to identify p, q and σ with $p>0$ and $\sigma>0$ that rewrite this equation as

$$\boxed{\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0,}$$

Since T_0 does not depend on x, then we can move it through the first derivative and rearrange the terms to obtain

$$\frac{d}{dx}\left(T_0 \phi'(x)\right) + \alpha(x)\phi(x) + \lambda \rho_0(x)\phi(x) = 0$$

From this it is clear that

$$p(x) = T_0, \quad q(x) = \alpha(x) \quad \text{and} \quad \sigma(x) = \rho_0(x).$$

Recalling that $T_0 > 0$ and the density $\rho_0(x) > 0$ finishes the identification.

***5.3.3.** Consider the non-Sturm–Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm–Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

Given $\alpha(x), \beta(x)$, and $\gamma(x)$, what are $p(x), \sigma(x)$, and $q(x)$?

Multiplying yields

$$H\frac{d^2\phi}{dx^2} + H\alpha\frac{d\phi}{dx} + (H\lambda\beta + H\gamma)\phi = 0.$$

To make the second term go away we need

$$\frac{dH\phi'}{dx} = H\frac{d^2\phi}{dx^2} + H\alpha\frac{d\phi}{dx}$$

Since

$$\frac{dH\phi'}{dx} = H'\phi' + H\frac{d^2\phi}{dx^2}$$

this implies $H' = H\alpha$. Consequently

$$\int_{H_0}^{H(x)} \frac{dH}{H} = \int_0^x \alpha(x)\,dx$$

$$\log H(x) - \log H_0 = \int_0^x \alpha(x)\,dx$$

or

$$H(x) = H_0\, e^{\int_0^x \alpha(x)\,dx}$$

With this choice of H we obtain

$$\frac{d\,H g'}{dx} + (H\lambda\beta + H\gamma)\,y = 0.$$

which is the standard Sturm-Liouville form where

$$p(x) = H(x) = H_0\, e^{\int_0^x \alpha(x)\,dx}$$

$$r(x) = H(x)\beta(x) = H_0\,\beta(x)\, e^{\int_0^x \alpha(x)\,dx}$$

and

$$q(x) = H(x)\,\gamma(x) = H_0\,\gamma(x)\, e^{\int_0^x \alpha(x)\,dx}.$$