HW6 (extra) due Friday, May 3

Turn in 12.6#8eg

Practice 5.3#4, 5.3#8, 12.2.5#bd, 12.3#3, 12.6#7a, 12.6#9a

**5.3.4.** Consider heat flow with convection (see Exercise 1.5.2):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - V_0\frac{\partial u}{\partial x}.$$

**(a)** Show that the spatial ordinary differential equation obtained by separation of variables is not in Sturm–Liouville form.

For separation of variables write

$$u(x,t) = \phi(x)h(t)$$

Then

$$\phi(x)h'(t) = k\phi''(x)h(t) - V_0\phi'(x)h(t)$$

and

$$\frac{h'(t)}{h(t)} = \frac{k\phi''(x) - V_0\phi'(x)}{\phi(x)} = -\lambda$$

Consequently,

$$h'(t) = -\lambda h(t) \quad \text{and} \quad \phi''(x) - \frac{V_0}{k}\phi'(x) + \frac{\lambda}{k}\phi(x) = 0$$

The equation in $\phi''$ is not in Sturm-Liouville form.

Note that this equation can be put in Sturm-Liouville form by multiplying it by

$$H(x) = H_0 e^{-\int_0^x \frac{V_0}{k}dx} = H_0 e^{-V_0 x/k}.$$

Thus $H'(x) = \frac{-V_0}{k}H(x)$ implies

$$\frac{d}{dx}\left(H(x)\phi'(x)\right) + \frac{\lambda}{K} H(x)\phi(x) = 0$$

which is in Sturm-Liouville form with

$$p(x) = H(x) = H_0 e^{-V_0 x/K}$$

and $\sigma(x) = \dfrac{H(x)}{K} = \dfrac{H_0}{K} e^{-V_0 x/K}$.

*(b) Solve the initial boundary value problem

$$u(0,t) = 0$$
$$u(L,t) = 0$$
$$u(x,0) = f(x).$$

It's easier to solve the original ODE directly. So consider

$$K\phi''(x) - V_0\phi'(x) + \lambda\phi(x) = 0 \quad \text{with } \phi(0) = 0 \text{ and } \phi(L) = 0$$

Substituting $e^{rx}$ yields

$$Kr^2 - V_0 r + \lambda = 0$$

consequently, by the quadratic formula

$$r = \frac{V_0 \pm \sqrt{V_0^2 - 4K\lambda}}{2K}$$

If $v_0^2 > 4k\lambda$ then the solution is given by exponentials in which case

$$g(x) = Ae^{r_1 x} + Be^{r_2 x}$$

and it is impossible for a non-zero solution to satisfy the boundary conditions.

Similarly if $v_0^2 = 4k\lambda$ no non-zero solution satisfies the boundary conditions.

Therefore, $v_0^2 < 4k\lambda$ and the solution is

$$g(x) = e^{\frac{v_0}{2k}x}\left( A\cos\frac{\sqrt{4k\lambda - v_0^2}}{2k}x + B\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x \right)$$

Since

$$g(0) = A = 0$$

and

$$g(L) = e^{\frac{v_0}{2k}L} B\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}L = 0$$

it follows that $\frac{\sqrt{4k\lambda - v_0^2}}{2k}L = n\pi$ for $n = 1, 2, \ldots$

Consequently $4k\lambda - v_0^2 = \left(\frac{2kn\pi}{L}\right)^2$

or $\lambda = k\left(\frac{n\pi}{L}\right)^2 + \frac{v_0^2}{4k}$

Since solving for $h(t)$ yields that

$$h(t) = Ce^{-\lambda t}$$

the superposition principle implies

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t} e^{\frac{V_0}{2k}x} \sin\frac{n\pi x}{L}$$

where $\lambda_n = k\left(\frac{n\pi}{L}\right)^2 + \frac{V_0^2}{4k}$.

Since the ODE for $\phi$ can be put into Sturm-Liouville form we know that the eigenfunctions

$$e^{\frac{V_0}{2k}x} \sin\frac{n\pi x}{L}$$

are orthogonal relative the weight function

$$\sigma(x) = e^{-V_0 x/k}$$

where we have taken $H_0 = k$ for convenience.

Solving for the constants $B_n$ to satisfy the initial conditions using orthogonality then yields

$$u(x,0) = \sum_{n=1}^{\infty} B_n e^{\frac{V_0}{2k}x} \sin\frac{n\pi x}{L} = f(x)$$

so that

$$\sum_{n=1}^{\infty} B_n e^{\frac{V_0}{2k}x} \sin\frac{n\pi x}{L} e^{\frac{V_0}{2k}x} \sin\frac{m\pi x}{L} e^{-\frac{V_0}{k}x} = f(x) e^{\frac{V_0}{2k}x} \sin\frac{m\pi x}{L} e^{-\frac{V_0}{k}x}$$

or

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} = f(x)\, e^{-\frac{v_0}{2k}x} \sin\frac{m\pi x}{L}.$$

Integrating then yields

$$B_m = \frac{2}{h}\int_0^L f(x)\, e^{-\frac{v_0}{2k}x} \sin\frac{m\pi x}{L}\, dx$$

Note carefully the sign on the argument to the exponential comes from using the weight needed for orthogonality.

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t} e^{\frac{v_0}{2k}x} \sin\frac{n\pi x}{L}$$

where $\lambda_n = k\left(\frac{n\pi}{L}\right)^2 + \frac{v_0^2}{4k}$

and $B_n = \frac{2}{h}\int_0^L f(x)\, e^{-\frac{v_0}{2k}x} \sin\frac{n\pi x}{L}\, dx.$

**(c)** Solve the initial boundary value problem

$$\frac{\partial u}{\partial x}(0,t) = 0$$
$$\frac{\partial u}{\partial x}(L,t) = 0$$
$$u(x,0) = f(x).$$

Begin by solving

$$k\varphi''(x) - V_0\varphi'(x) + \lambda\varphi(x) = 0 \quad \text{with } \varphi'(0)=0 \text{ and } \varphi'(L)=0.$$

Again substituting $e^{rx}$ yields

$$r = \frac{V_0 \pm \sqrt{V_0^2 - 4k\lambda}}{2k}$$

In the case $V_0^2 = 4k\lambda$ we obtain

$$\varphi(x) = Ae^{\frac{V_0}{2k}x} + Bxe^{\frac{V_0}{2k}x}$$

Now $\varphi'(x) = \left(\frac{V_0}{2k}A + B\right)e^{\frac{V_0}{2k}x} + \frac{V_0}{2k}Bx\,e^{\frac{V_0}{2k}x}$

$$\varphi'(0) = \frac{V_0}{2k}A + B = 0 \qquad \text{implies} \quad B = -\frac{V_0}{2k}A$$

Then

$$\varphi'(L) = \frac{V_0}{2k}BL\,e^{\frac{V_0}{2k}L} = 0 \qquad \text{implies} \quad A=B=0$$

Therefore, the only eigenfunctions are again when $V_0^2 < 4k\lambda$

$$\varphi(x) = e^{\frac{V_0}{2k}x}\left(A\cos\frac{\sqrt{4k\lambda - V_0^2}}{2k}x + B\sin\frac{\sqrt{4k\lambda - V_0^2}}{2k}x\right).$$

Differentiating yields

$$\phi'(x) = \frac{v_0}{2k} e^{\frac{v_0}{2k}x}\left(A\cos\frac{\sqrt{4k\lambda - v_0^2}}{2k}x + B\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x\right)$$

$$+ \frac{\sqrt{4k\lambda - v_0^2}}{2k} e^{\frac{v_0}{2k}x}\left(-A\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x + B\cos\frac{\sqrt{4k\lambda - v_0^2}}{2k}x\right).$$

Thus

$$\phi'(0) = \frac{v_0}{2k}A + \frac{\sqrt{4k\lambda - v_0^2}}{2k}B = 0 \qquad \text{so} \quad A = -\frac{\sqrt{4k\lambda - v_0^2}}{v_0}B$$

consequently

$$\phi'(x) = \frac{v_0}{2k} e^{\frac{v_0}{2k}x} B\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x + \frac{4k\lambda - v_0^2}{2kv_0} e^{\frac{v_0}{2k}x} B\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x$$

$$= Be^{\frac{v_0}{2k}x}\left(\frac{v_0}{2k} + \frac{4k\lambda - v_0^2}{2kv_0}\right)\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}x$$

and

$$\phi'(L) = Be^{\frac{v_0}{2k}x}\left(\frac{v_0}{2k} + \frac{4k\lambda - v_0^2}{2kv_0}\right)\sin\frac{\sqrt{4k\lambda - v_0^2}}{2k}L = 0$$

implies $\frac{\sqrt{4k\lambda - v_0^2}}{2k}L = n\pi$

or $\quad \lambda = k\left(\frac{n\pi}{L}\right)^2 + \frac{v_0^2}{4k} \quad$ for $\quad n = 0, 1, 2, \ldots$

Note that $n=0$ now corresponds to a non-zero eigenfunction, the constant eigenfunction, so we keep it.

Therefore

$$A = -\frac{\sqrt{4k\lambda - v_0^2}}{v_0} B = -\frac{2k}{v_0} \cdot \frac{n\pi}{L} B$$

and

$$f(x) = B e^{\frac{v_0}{2k}x} \left( -\frac{2k}{v_0} \cdot \frac{n\pi}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi x}{L} \right).$$

Now we see that $n=0$ doesn't work. So by superposition

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t} e^{\frac{v_0}{2k}x} \left( -\frac{2k}{v_0} \cdot \frac{n\pi}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi x}{L} \right)$$

where $\lambda_n = k\left(\frac{n\pi}{L}\right)^2 + \frac{v_0^2}{4k}$.

Now use orthogonality with respect to the weight $e^{-\frac{v_0 x}{k}}$ to solve for the initial condition

$$u(x,0) = \sum_{n=1}^{\infty} B_n e^{\frac{v_0}{2k}x} \left( -\frac{2k}{v_0} \cdot \frac{n\pi}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi x}{L} \right) = f(x)$$

Orthogonality implies

$$B_n \left( -\frac{kn\pi}{v_0} + \frac{L}{2} \right) = \int_0^L f(x) e^{-\frac{v_0}{2k}x} \left( -\frac{2k}{v_0} \cdot \frac{n\pi}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi x}{L} \right) dx$$

Therefore the solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t} e^{\frac{v_0}{2k}x} \left( -\frac{2k}{v_0} \cdot \frac{n\pi}{L} \cos\frac{n\pi x}{L} + \sin\frac{n\pi x}{L} \right)$$

where $\lambda_n = k\left(\frac{n\pi}{L}\right)^2 + \frac{v_0^2}{4k}$ and

$$B_n = \frac{1}{-\frac{kn\pi}{v} + \frac{L}{2}} \int_0^L f(x) e^{-\frac{v_0}{2k}x} \left( -\frac{2k}{v_0} \cdot \frac{n\pi}{L} \cos\frac{n\pi x}{L} + \sin\frac{n\pi x}{L} \right) dx.$$

**5.3.8.** Show that $\lambda \geq 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(1) = 0.$$

Is $\lambda = 0$ an eigenvalue?

This is a Sturm-Liouville problem with

$p(x)=1$, $q(x) = -x^2$ and $\sigma(x) = 1$.

The Rayleigh quotient is

$$\lambda = \frac{\phi(x)\,\phi'(x)\Big|_0^1 + \int_0^1 \left(\phi'(x)^2 + x^2 \phi(x)^2\right) dx}{\int_0^1 \phi(x)^2\, dx}$$

$$= \frac{\int_0^1 \left(\phi'(x)^2 + x^2 \phi(x)^2\right) dx}{\int_0^1 \phi(x)^2\, dx}$$

Since $\phi(x) \neq 0$ then $\int_0^1 x^2 \phi(x)^2\, dx > 0$. It follows that

$$\int_0^1 \left(\phi'(x)^2 + x^2 \phi(x)^2\right) dx > 0$$

and consequently that $\lambda > 0$.

Therefore $\lambda = 0$ is not an eigenvalue.

**12.2.5.** Solve using the method of characteristics (if necessary, see Section 12.6):

*(b) $\frac{\partial w}{\partial t} + x\frac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

Let $x = x(t)$. Then

$$\frac{d}{dt} w(x(t),t) = \frac{\partial w}{\partial t} + x'(t)\frac{\partial w}{\partial x}$$

It follows that the ODE for the characteristics is

$$x'(t) = x(t)$$

and along the characteristics the PDE becomes

$$\frac{d}{dt} w(x(t),t) = 1$$

Therefore

$$x(t) = x_0 e^t \quad \text{and} \quad w(x(t),t) = t + w(x_0,0) = t + f(x_0)$$

It follows that the solution along the characteristics is

$$w(x_0 e^t, t) = t + f(x_0)$$

Solving $x = x_0 e^t$ for $x_0$ then yields $x_0 = xe^{-t}$ so

$$w(x,t) = t + f(xe^{-t}).$$

*(d) $\frac{\partial w}{\partial t} + 3t\frac{\partial w}{\partial x} = w$ with $w(x,0) = f(x)$

Let $x = x(t)$. Then $\frac{dw(x(t),t)}{dt} = \frac{\partial w}{\partial t} + x'(t)\frac{\partial w}{\partial x}$. Therefore the ODEs are

$$x'(t) = 3t \quad \text{and} \quad \frac{d}{dt} w(x(t),t) = w(x(t),t)$$

It follows that

$$x(t) = \frac{3}{2}t^2 + x_0 \quad \text{and} \quad w(x(t),t) = w(x_0,0)\,e^t = f(x_0)\,e^t$$

The solution along the characteristics is

$$w\left(\frac{3}{2}t^2 + x_0, t\right) = f(x_0)\,e^t$$

Setting $x = \frac{3}{2}t^2 + x_0$ so that $x_0 = x - \frac{3}{2}t^2$ yields

$$w(x,t) = f\left(x - \frac{3}{2}t^2\right)e^t$$

**12.3.3.** An alternative way to solve the one-dimensional wave equation (12.3.1) is based on (12.3.2) and (12.3.3). Solve the wave equation by introducing a change of variables from $(x, t)$ to two moving coordinates $(\xi, \eta)$, one moving to the left (with velocity $-c$) and one moving to the right (with velocity $c$):

$$\xi = x - ct \qquad \text{and} \qquad \eta = x + ct.$$

For reference,

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,} \tag{12.3.1}$$

$$w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = P(x - ct). \tag{12.3.2}$$

and

$$v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = Q(x + ct). \tag{12.3.3}$$

Since $\xi = x - ct$ and $\eta = x + ct$ then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial u}{\partial \xi}(-c) + \frac{\partial u}{\partial \eta} c$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial \xi}(-c) + \frac{\partial u}{\partial \eta} c\right) = \frac{\partial}{\partial \xi}\frac{\partial u}{\partial t}(-c) + \frac{\partial}{\partial \eta}\frac{\partial u}{\partial t} c$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}(-c) + \frac{\partial u}{\partial \eta} c\right)(-c) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}(-c) + \frac{\partial u}{\partial \eta} c\right)(c)$$

$$= c^2 \frac{\partial^2 u}{\partial \xi^2} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \frac{\partial u^2}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial u^2}{\partial \eta^2}.$$

Therefore, substituting yields

$$\cancel{c^2 \frac{\partial^2 u}{\partial \xi^2}} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{c^2 \frac{\partial^2 u}{\partial \eta^2}} - c^2 \left( \cancel{\frac{\partial^2 u}{\partial \xi^2}} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta^2}} \right) = 0.$$

$$-\cancel{2c^2} \frac{\partial^2 u}{\partial \xi \partial \eta} - \cancel{2c^2} \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

Thus $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$.

It follows $\frac{\partial u}{\partial \xi} = \int \frac{\partial^2 u}{\partial \xi \partial \eta} \, d\eta = h(\xi)$ since $\xi$ is constant when doing partial differentiation with respect to $\eta$.

and so $u = \int \frac{\partial u}{\partial \xi} \, d\xi = \int h(\xi) \, d\xi + G(\eta) = F(\xi) + G(\eta)$ where $F'(\xi) = h(\xi)$ and $G(\eta)$ is the constant of integration since $\eta$ is constant when doing partial differentiation with respect to $\xi$. Therefore

$$u(x,t) = F(x-ct) + G(x+ct).$$

**12.6.7.** Solve the following problems:

(a) $\dfrac{\partial \rho}{\partial t} + \rho^2 \dfrac{\partial \rho}{\partial x} = 0, \qquad \rho(x,0) = \begin{cases} 3 & x < 0 \\ 4 & x > 0 \end{cases}$

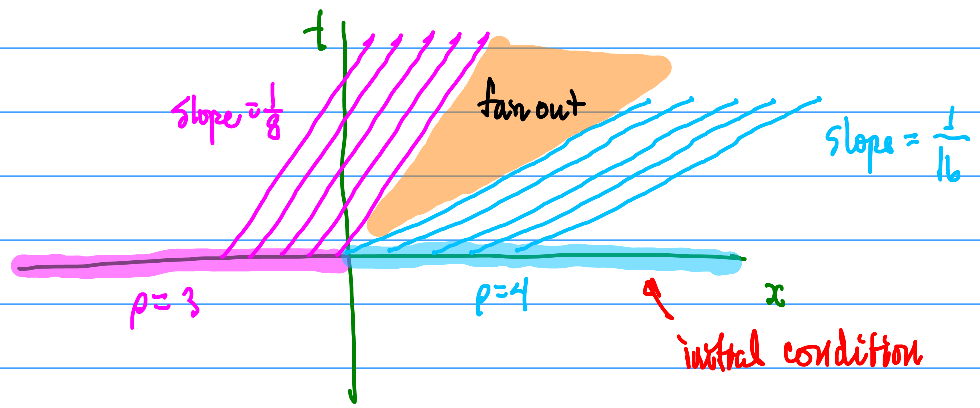
Let $x = x(t)$ an consider

$$\frac{d}{dt}\rho(x(t),t) = \frac{\partial \rho}{\partial t} + x'(t)\frac{\partial \rho}{\partial x}$$

Therefore

$$x'(t) = \rho^2(x(t),t) \qquad \text{in which case} \qquad \frac{d}{dt}\rho(x(t),t) = 0$$

The second ODE implies $\rho$ is constant along characteristics. Thus, $\rho(x(t),t) = \rho(x_0,0)$. Then $x'(t) = \rho^2(x_0,0)$ implies $x(t) = \rho^2(x_0,0)t + x_0$.




In the fan out region we imagine a family of characteristics with $\rho \in [3,4]$ starting from $x_0 = 0$. In the region

$$x = \rho^2 t \quad \text{or} \quad \rho = \sqrt{\frac{x}{t}}.$$

Thus

$$\rho(x,t) = \begin{cases} 3 & \text{if} \quad x \leq 9t \\ \sqrt{\dfrac{x}{t}} & \text{if} \quad 9t < x < 16t \\ 4 & \text{if} \quad x \geq 16t. \end{cases}$$

**12.6.8.** Solve subject to the initial condition $\rho(x,0) = f(x)$:

*(e) $\dfrac{\partial \rho}{\partial t} - t^2 \dfrac{\partial \rho}{\partial x} = -\rho$

Let $x = x(t)$ so that

$$\frac{d}{dt}\rho(x(t),t) = \frac{\partial \rho}{\partial t} + x'(t)\frac{\partial \rho}{\partial t}$$

This leads to the ODEs

$$x'(t) = -t^2 \qquad \text{and} \qquad \frac{d}{dt}\rho(x(t),t) = -\rho.$$

Therefore

$$x(t) = -\int t^2\,dt = -\frac{1}{3}t^3 + x_0$$

and

$$\rho(x(t),t) = \rho(x_0,0)\,e^{-t} = f(x_0)\,e^{-t}$$

Along the characteristics the solution is

$$\rho\left(-\frac{1}{3}t^3 + x_0, t\right) = f(x_0)\,e^{-t}$$

Setting $x = -\frac{1}{3}t^3 + x_0$ so that $x_0 = x + \frac{1}{3}t^3$ yields

$$\rho(x,t) = f\left(x + \frac{1}{3}t^3\right)e^{-t}.$$

*(g) $\frac{\partial \rho}{\partial t} + x\frac{\partial \rho}{\partial x} = t$

Let $x = x(t)$ so that

$$\frac{d}{dt}\rho(x(t),t) = \frac{\partial \rho}{\partial t} + x'(t)\frac{\partial \rho}{\partial x}$$

This leads to the ODEs

$$x'(t) = x(t) \qquad \text{and} \qquad \frac{d}{dt}\rho(x(t),t) = t$$

Solving for the characteristics yields

$$x(t) = x_0 e^{t}$$

Then

$$\rho(x(t),t) = \int t\,dt = \frac{1}{2}t^2 + \rho(x_0,0) = \frac{1}{2}t^2 + f(x_0)$$

Therefore along the characteristics

$$\rho(x_0e^{t}, t) = \frac{1}{2}t^2 + f(x_0)$$

Set $x = x_0e^{t}$ so that $x_0 = xe^{-t}$. Then

$$\rho(x,t) = \frac{1}{2}t^2 + f(xe^{-t}).$$

**12.6.9.** Determine a parametric representation of the solution satisfying $\rho(x,0) = f(x)$:

*(a) $\dfrac{\partial \rho}{\partial t} - \rho^2 \dfrac{\partial \rho}{\partial x} = 3\rho$

Let $x = x(t)$. Then

$$\frac{d}{dt} \rho(x(t), x) = \frac{\partial \rho}{\partial t} + x'(t) \frac{\partial \rho}{\partial x}.$$

This yields to ODEs

$$x'(t) = -\rho^2 \qquad \text{and} \qquad \frac{d}{dt} \rho(x(t), t) = \rho(x(t), t).$$

Soling the second ODE first gives

$$\rho(x(t), t) = \rho(x_0, 0)\, e^{t} = f(x_0)\, e^{t}$$

Substituting into the equation for the characteristics obtains

$$x'(t) = -\left(f(x_0)\, e^{t}\right)^2 = -\left(f(x_0)\right)^2 e^{2t}$$

so

$$x(t) = -\frac{1}{2} \left(f(x_0)\right)^2 e^{2t} =$$

A parametric representation of the solution is given along the characteristics as

$$\rho\left(-\frac{1}{2}\left(f(x_0)\right)^2 e^{2t}, t\right) = f(x_0)\, e^{t},$$