PDE. $\cancel{\frac{\partial u}{\partial t}} = k\frac{\partial^2 u}{\partial x^2}$ — constant density and const. heat capacity

initial cond. $u(x,0) = f(x)$ for $x \in [0,L]$

boundary cond. $\frac{\partial u}{\partial x}(0,t) = 0$ $\frac{\partial u}{\partial x}(L,t) = 0$

conservation of total energy

$$\text{total energy} = \int_0^L e(x,t)\,dx = \int_0^L c(x)\,\rho(x)\,u(x,t)\,dx$$

const not depend on x

$$\text{total energy} = c\rho\int_0^L u(x,t)\,dx$$ in homogeneous bar

Differentiate w.r.t time is zero if total energy conserved

$$\frac{d}{dt}\int_0^L u(x,t)\,dx = \int_0^L \frac{\partial}{\partial t}u(x,t)\,dx = \int_0^L k\frac{\partial^2 u(x,t)}{\partial x^2}\,dx$$

by the Fundamental Theorem of Calculus

$$= k\frac{\partial u(L,t)}{\partial x} - k\frac{\partial u(0,t)}{\partial x} = 0 - 0 = 0$$

ODE $\frac{d^2u}{dx^2} = 0$, $u'(0) = 0$, $u'(L) = 0$

$$\frac{du}{dx} = \int \frac{d^2u}{dx^2}\,dx = \int 0\,dx = C$$

$$u = \int \frac{du}{dx}\,dx = \int C\,dx = Cx + D$$

$u'(0) = C = 0$

$u'(L) = C = 0$ what is D?

equilibrium state

$u(x) = Cx + D = D$

the state at $t \to \infty$.

$$\int_0^L u(x,0)\,dx = \int_0^L D\,dx$$

$$\int_0^L f(x)\,dx = LD$$

$$D = \frac{1}{L}\int_0^L f(x)\,dx.$$

average temperature..

Here are some problems to practice in preparation for the quiz of Wednesday.

- Exercise 1.4.1a-h.
- Exercise 1.4.7a-c.

1.4.1 # * (f) $\frac{Q}{K_0} = x^2, \quad u(0) = T, \quad \frac{\partial u}{\partial x}(L) = 0$

heat capacity, density, temperature, position in the rod or material, time, conductivity, internal production of heat energy in the rod.

$$c(x)\,\rho(x)\,\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + Q(x,t)$$

$$c\rho\,\cancel{\frac{\partial u}{\partial t}} = K_0\frac{\partial^2 u}{\partial x^2} + Q$$

equilibrium state
no time

$$0 = \frac{d^2u}{dx^2} + \frac{Q}{K_0} = \frac{d^2u}{dx^2} + x^2$$

Thus

$$u'' = -x^2 \qquad u(0) = T \qquad u'(L) = 0$$

$$u'(x) = -\int x^2\,dx = -\frac{1}{3}x^3 + C$$

$$u(x) = \int\left(-\frac{1}{3}x^3 + C\right)dx = -\frac{1}{12}x^4 + Cx + D$$

Now use boundary conditions to solve for C and D

$$u(0) = -\frac{1}{2}0^4 + C\cdot 0 + D = T \qquad \text{so} \qquad D = T$$

$$u'(L) = -\frac{1}{3}L^3 + C = 0 \qquad \text{so} \qquad C = \frac{1}{3}L^3$$

Answer:

$$u(x) = -\frac{1}{12}x^4 + \frac{1}{3}L^3\,x + T$$

S compact and $S = \overline{S^{\text{int}}}$

Greens Theorem: Let S be a regular region with piecewise smooth boundary.

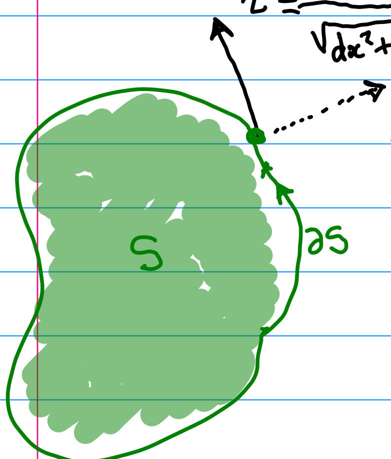
$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

proof apply fundamental theorem of calculus from 1D calculus

use partition of unity + implicit function theorem.

$$\int_{\partial S} P\,dx + Q\,dy = \int_{\partial S} \underbrace{\begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}}_{\text{like tangent}} = \int_{\partial S} \underbrace{\begin{bmatrix} Q \\ -P \end{bmatrix}}_{F} \cdot \underbrace{\begin{bmatrix} dy \\ -dx \end{bmatrix}}_{\text{like the normal}}$$

$$\hat{t} = \frac{(dx, dy)}{\sqrt{dx^2 + dy^2}}$$



$$\hat{n} = \frac{(dy, -dx)}{\sqrt{dx^2 + dy^2}}$$

$$= \int_{\partial S} F \cdot \hat{n} \sqrt{dx^2 + dy^2}$$

$$= \int_{\partial S} F \cdot \hat{n}\, d\tau$$

$d\tau$ ← arc length.

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_S \nabla \cdot F \, dA$$

Writing Green's theorem using the divergence...

$$\iint_S \nabla \cdot F \, dA = \int_{\partial S} F \cdot \hat{n}\, d\tau$$

generalizes to higher dimensions...

$$\iiint_R \nabla \cdot F \, dV = \iint_{\partial R} F \cdot \hat{n}\, dS$$