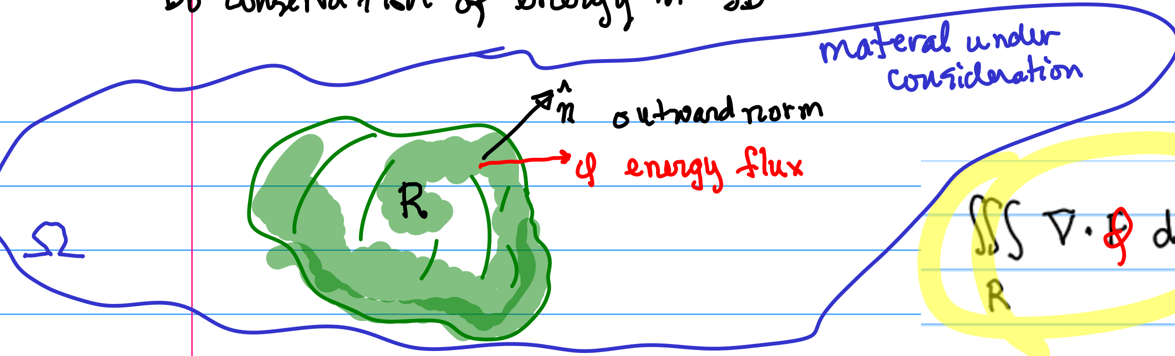
Do conservation of energy in 3D




$$\iiint_R \nabla \cdot \phi \, dV = \iint_{\partial R} \phi \cdot \hat{n} \, dS$$

| | | | | |
|---|---|---|---|---|
| Rate of change of heat energy | = | heat energy flowing across the boundaries per unit time | + | heat energy generated inside per unit time, |

$$\frac{d}{dt}\iiint_R e(\vec{x},t)\, dV = -\iint_{\partial R} \phi \cdot \hat{n}\, dS + \iiint_R Q(\vec{x},t)\, dV$$

$\partial R$ ← surface of R

$$\frac{d}{dt}\iiint_R e(\vec{x},t)\, dV = -\iiint_R \nabla \cdot \phi \, dV + \iiint_R Q(\vec{x},t)\, dV$$

$$\iiint_R \left( \frac{\partial}{\partial t} e(\vec{x},t) + \nabla \cdot \phi - Q(\vec{x},t) \right) dV = 0$$

If the integral is zero for all different regions R inside $\Omega$ the material under consideration, then

PDE $\quad \frac{\partial}{\partial t} e(\vec{x},t) + \nabla \cdot \phi - Q(\vec{x},t) = 0 \quad$ for all $\vec{x} \in \Omega$

Convert to temperature $\quad e(\vec{x},t) = c(\vec{x})\rho(\vec{x})\, u(\vec{x},t)$

$$\phi(x,t) = -K_0(\vec{x}) \nabla u(\vec{x},t)$$

Heat eq. in 3D.

$$c(\vec{x})\rho(\vec{x}) \frac{\partial u(\vec{x},t)}{\partial t} = \nabla \cdot \left( K_0(\vec{x}) \nabla u(\vec{x},t) \right) + Q(x,t)$$

# CHAPTER 2

# Method of Separation of Variables

Consider heat in a 1D rod:

$$\underbrace{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho}}_{\text{linear PDE ... I.e. } u \text{ only appears linearly.}}, \qquad \begin{array}{l} t > 0 \\ 0 < x < L, \end{array}$$

Let $L(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2}$ and $f(x,t) = \frac{Q(x,t)}{c\rho}$

Now the heat equation looks like

$$L(u) = f$$

looks like $Ax=b$ in linear algebra...

$Ax=0$ homo-geneous

Note $L(c_1u + c_2v) = c_1L(u) + c_2L(v)$ ← linearity.

If $f=0$ ← homogeneous problem.

Then $\underbrace{L(u)=0 \text{ and } L(v)=0 \text{ implies } L(c_1u+c_2v)=0}_{\text{This is called the superposition principle.}}$

**Principle of Superposition**
If $u_1$ and $u_2$ satisfy a linear *homogeneous* equation, then an arbitrary linear combination of them, $c_1u_1 + c_2u_2$, also satisfies the same linear homogeneous equation.

Also consider boundary conditions.

Heat bath → $u(0,t) = f(t)$

prescribed flux through the boundary → $\frac{\partial u}{\partial x}(L,t) = g(t)$

insulating → $\boxed{\frac{\partial u}{\partial x}(0,t) = 0}$ ← homogeneous.

Newton's law of cooling → $-K_0 \frac{\partial u}{\partial x}(L,t) = h[u(L,t) - g(t)]$.

temperature outside

Notice u appears linearly in any of the boundary conditions,

**Question**: if u and v both satisfy the boundary condition does $c_1 u + c_2 v$ also satisfy the same boundary condition?

Means homogeneous boundary

Examples: Heat bath with zero temp. $u(0,t) = 0$

Insulating $\frac{\partial u}{\partial x}(0,t) = 0$

Newton with zero temp. $-K_0 \frac{\partial u}{\partial x}(L,t) = h(u(L,t) - 0)$

Idea: To solve a problem with inhomogeneous boundary, shift the temperature and then solve the homogeneous boundary.

Details?, Later...

PDE: homogeneous $\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}}$ $\quad 0 < x < L$, $t > 0$

BC: homogeneous $\boxed{\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}}$

We'll solve this first.

IC: not homogeneous... $\boxed{u(x,0) = f(x).}$