PDE: homogeneous $\boxed{\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}}$ $\quad 0 < x < L,\ t > 0$

BC: homogeneous $\boxed{\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}}$

Will solve this first.

IC: not homogeneous $\boxed{u(x,0) = f(x).}$

Idea: Look for simple (separable) solutions to the homogeneous PDE that satisfies the homogeneous boundary condition. Then use the superposition priciple to create a linear combination of seperable solutions that satisfy the initial condition.

Let $u(x,t) = \phi(x)\,G(t)$ plug it in...

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \phi(x)\,G'(t) = k\,\phi''(x)\,G(t)$$

now put all t's on one side and all x's on the other.

$$\frac{\partial u}{\partial t} = \frac{\partial\, \phi(x)G(t)}{\partial t} = \phi(x)\,G'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2\, \phi(x)G(t)}{\partial x^2} = \phi''(x)\,G(t)$$

Therefore

$$\underbrace{\frac{G'(t)}{kG(t)}}_{\substack{\text{only time} \\ \text{(no dependency on } x)}} = \underbrace{\frac{\phi''(x)}{\phi(x)}}_{\substack{\text{only space} \\ \text{(no dependency on } t)}} = \underbrace{-\lambda}_{\substack{\text{constant} \\ \text{(neither dependent on } t \text{ or } x)}}$$

Two ODEs

$$G'(t) = -\lambda k G(t) \quad \text{and} \quad \phi''(x) = -\lambda \phi(x)$$

not homogeneous..

IC: $\boxed{u(x,0) = f(x).}$

homogeneous

BC: $\boxed{\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}}$

I'm ignoring the IC for now

because it's too much to hope such a simple solution could also satisfy the boundary conditions..

$\phi(0)=0$, $\phi(L)=0$

Since the initial condition will be solved for using a superposition of separable solutions later. Do the x equation first..

$$\phi''(x) = -\lambda \phi(x), \quad \phi(0)=0, \quad \phi(L)=0$$

Solve: What is λ?

Case $\lambda=0$: Then $\phi''(x)=0$

General solution $\phi(x) = Cx + D$

Satisfy B.C. $\phi(0) = D = 0$ so $D=0$

$\phi(L) = CL + 0 = CL = 0$ so $C=0$

Then $\phi(x) = 0$

$u(x,t) = \phi(x)\,G(t) = 0$

Since I can't build a non-zero initial condition out of a superposition of zero solutions $\lambda=0$ is useless...

<u>Case</u> $\lambda < 0$ Then $\varphi''(x) = |\lambda| \varphi(x)$ since $-\lambda = |\lambda|$ assume $\lambda < 0$

general solution $\varphi(x) = c_1 e^{\sqrt{|\lambda|}\, x} + c_2 e^{-\sqrt{|\lambda|}\, x}$.

note that this solution satisfies the ODE. also, since it's a second order ODE the two constants c_1 and c_2 mean we have the general solution. Remark need to know $e^{\sqrt{|\lambda|}x}$ and $e^{-\sqrt{|\lambda|}x}$ are linearly independent. Use the Wronskian in ODE class...

Satisfy B.C. $\varphi(0) = c_1 + c_2 = 0 \qquad c_2 = -c_1$

$$\varphi(L) = c_1 e^{\sqrt{|\lambda|} L} - c_1 e^{-\sqrt{|\lambda|} L} = 0$$

$$2 c_1 \underbrace{\left(\frac{e^{\sqrt{|\lambda|} L} - e^{-\sqrt{|\lambda|} L}}{2} \right)}_{\sinh(\sqrt{|\lambda|} L) \neq 0} = 0$$

therefore $c_1 = 0$

Then $\varphi(x) = 0$ not useful.

<u>Case</u> $\lambda > 0$ then $\varphi''(x) = -|\lambda| \varphi(x)$

general solution $\varphi(x) = A \cos(\sqrt{|\lambda|}\, x) + B \sin(\sqrt{|\lambda|}\, x)$

B.C. $\varphi(0) = A \cos 0 + B \sin 0 = A = 0$

$\varphi(L) = B \sin(\sqrt{|\lambda|}\, L) = 0$

therefore either $B = 0$ or $\sin(\sqrt{|\lambda|}\, L) = 0$

The $B = 0$ gives $\varphi(x) = 0$ which is not useful.

Suppose $\sin(\sqrt{|\lambda|}\,L) = 0$ then $\sqrt{|\lambda|}\,L = n\pi$ for any $n \in \mathbb{Z}$

or $\sqrt{|\lambda|} = \frac{n\pi}{L}$

solution $\varphi(x) = B\sin\left(\frac{n\pi x}{L}\right)$ for $n \in \mathbb{Z}$

if $n = 0$ then $\varphi(x) = 0$ not useful.

if $n < 0$ then $\varphi(x) = B\sin\left(-\frac{|n|\pi x}{L}\right) = -B\sin\left(\frac{|n|\pi x}{L}\right)$

So I get all posible non-zero solutions as

$$\varphi_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } n = 1, 2, 3, 4, \ldots$$

Now solve the ODE in time

$$G'(t) = -\lambda k G(t) = -\left(\frac{n\pi}{L}\right)^2 k\, G(t)$$

solution $G_n(t) = A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}$

Therefore

$$u_n(x,t) = \varphi_n(x) G_n(t) = B_n \sin\left(\frac{n\pi x}{L}\right) A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$u_n(x,t) = a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

Write

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

Note u satisfies the PDE and BC because they were homogeneous. Now solve for a_n so that $u(x,0) = f(x)$ satisfies the initial condition.