Laplace equation on a rectangle.

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$u(x,0) = 0 \quad u(x,H) = f_2(x)$

$u(0,y) = 0 \quad u(W,y) = 0$

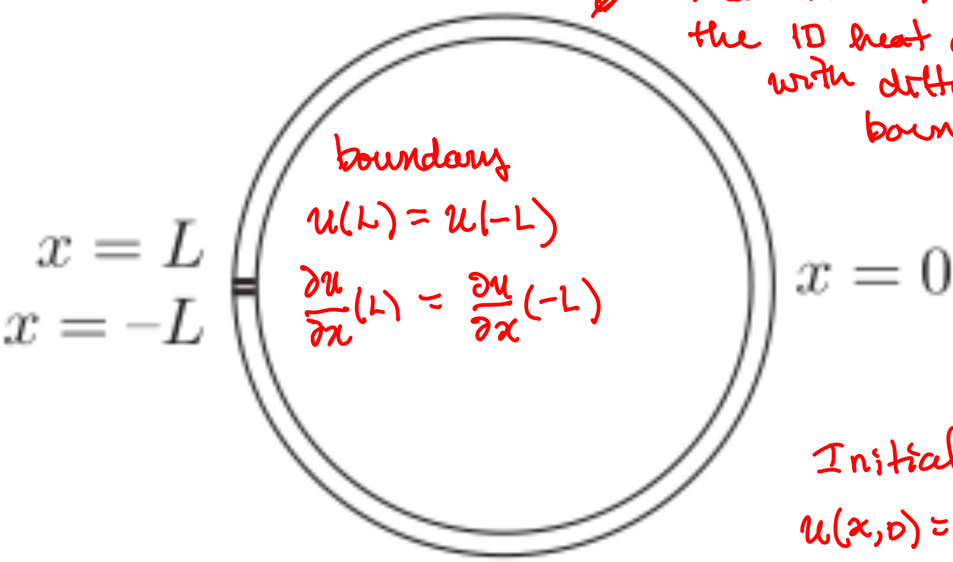
Super position of seperable solutions

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{W} \sinh\frac{n\pi y}{W}$$

By orthogonality

$$B_m = \frac{2}{W \sinh\frac{m\pi H}{W}} \int_0^W \sin\frac{m\pi x}{W} f_2(x)\, dx$$

Heat equation on a circle



assume the rod is so thin that this is the same as the 1D heat equation but with different boundary

boundary

$u(L) = u(-L)$

$\frac{\partial u}{\partial x}(L) = \frac{\partial u}{\partial x}(-L)$

Initial

$u(x,0) = f(x)$

$x \in [-L, L]$

Note defining $x \in [-L, L]$ instead of $[0, L]$ just makes the algebra easier...

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad u(x,t) = \varphi(x)\,G(t)$$

look for separable solns.

$$\varphi(x)\,G'(t) = k\,\varphi''(x)\,G(t)$$

$$\frac{\varphi''(x)}{\varphi(x)} = \frac{G'(t)}{kG(t)} = -\lambda$$

ODEs

$$\varphi'' = -\lambda\varphi \qquad\qquad G' = -\lambda k G$$

$$\boxed{\varphi(-L) = \varphi(L)} \qquad\qquad G(0) = \text{ solve using superposition..}$$

$$\boxed{\varphi'(-L) = \varphi'(L)}$$

Case $\lambda > 0$

General solution: $\varphi(x) = c_1\cos(\sqrt{\lambda}\,x) + c_2\sin(\sqrt{\lambda}\,x)$

B.C.:

$$\varphi(-L) = c_1\cancel{\cos}(\sqrt{\lambda}\,L) - c_2\sin(\sqrt{\lambda}\,L) = \varphi(L) = c_1\cancel{\cos}(\sqrt{\lambda}\,L) + c_2\sin(\sqrt{\lambda}\,L)$$

Therefore $\quad 2c_2\sin(\sqrt{\lambda}\,L) = 0$

is $c_2 = 0$ or $\sin(\sqrt{\lambda}\,L) = 0$

before deciding check other B.C.

$$\varphi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$$

B.C.

$$\varphi'(-L) = c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,L) + c_2\sqrt{\lambda}\cancel{\cos}(\sqrt{\lambda}\,L) = \varphi'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,L) + c_2\sqrt{\lambda}\cancel{\cos}(\sqrt{\lambda}\,L)$$

Therefore $\quad 2c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,L) = 0$

is $c_1 = 0$ or $\sin(\sqrt{\lambda}\,L) = 0$

If $c_1 = 0$ and $c_2 = 0$ then $\varphi(x) = 0$ and I can't make a superposition out of that

If $\sin\sqrt{\lambda}\,L = 0$ then c_1 and c_2 can be anything.

Thus $\sin(\sqrt{\lambda}L)=0$ so $\sqrt{\lambda}L = n\pi$ for $n=1,2,\ldots$

so $\sqrt{\lambda} = \frac{n\pi}{L}$ or $\lambda = \left(\frac{n\pi}{L}\right)^2$.

So $\varphi(x) = c_1 \cos\left(\frac{n\pi}{L}x\right) + c_2 \sin\left(\frac{n\pi}{L}x\right)$ $\quad n=1,2,\ldots$

is a family of non-zero eigenfunctions... (for $\lambda>0$).

<u>Case $\lambda=0$:</u> $\varphi''=0$ with $\varphi(-L)=\varphi(L)$ $\varphi'(-L)=\varphi'(L)$

general solution $\varphi(x) = c_1 x + c_2$, $\varphi'(x) = c_1$

$\varphi(-L) = -c_1 L + \cancel{c_2} = \varphi(L) = c_1 L + \cancel{c_2}$

$2c_1 L = 0$ so $c_1 = 0$.

$\varphi'(-L) = 0 = \varphi'(L) = 0$ (nothing to solve for)

One more eigenfunction $\varphi(x) = c_2$

Case $\lambda < 0$.

eigenfunctions, $\sin n\pi x/L$ and $\cos n\pi x/L$. Not surprisingly, it can be shown that there are no eigenvalues in which $\lambda < 0$.

or eigenfunctions. ? Check this at home...

Now solve the other ODE

$G' = -\lambda k G$

general solution $G(t) = c_1 e^{-\lambda k t}$

thus $G_n(t) = e^{-\left(\frac{n\pi}{L}\right)^2 k t}$

Superposition to solve for the initial condition

$$u(x,t) = a_0 G_0(t) + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} G_n(t) + b_n \sin\frac{n\pi x}{L} G_n(t)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

Initial $\quad u(x,0) = f(x) \qquad x \in [-L, L]$

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) = f(x)$$

mult by the the different eigenfunction.

$1 \qquad \sin\frac{m\pi x}{L} \quad$ and $\quad \cos\frac{m\pi x}{L} \qquad m = 1, 2, \ldots$

and then integrate over $[-L, L]$ and use orthogonality.

Normalization

$$\int_{-L}^{L} 1^2\, dx = 2L, \qquad \int_{-L}^{L} \left(\sin\frac{m\pi x}{L}\right)^2 dx = \frac{2L}{2} = L, \qquad \int_{-L}^{L} \left(\cos\frac{m\pi x}{L}\right)^2 dx = L$$

trigonometry or integration by parts

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} \left(\sin\frac{n\pi x}{L}\right) f(x)\, dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} \left(\cos\frac{n\pi x}{L}\right) f(x)\, dx.$$