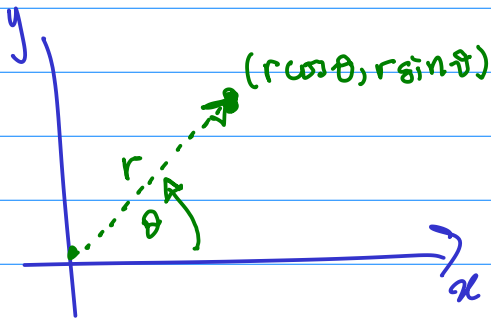


Polar Coordinates.



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\frac{\partial x}{\partial r} = \frac{\partial (r \cos \theta)}{\partial r} = \cos \theta$$

$$\frac{\partial y}{\partial r} = \frac{\partial (r \sin \theta)}{\partial r} = \sin \theta$$

$\frac{\partial}{\partial x}$ | holding y constant

$\frac{\partial}{\partial r}$ | holding θ constant.

Consider a function u defined on the plane...

solve system for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} r \cos \theta$$

$$r \cos \theta \frac{\partial u}{\partial r} = r \cos \theta \left(\frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \right)$$

subtract

$$\sin \theta \frac{\partial u}{\partial \theta} = \sin \theta \left(\frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} r \cos \theta \right)$$

$$r \cos \theta \frac{\partial u}{\partial r} - \sin \theta \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} (r \cos^2 \theta + r \sin^2 \theta)$$

similarly

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta$$

$$\frac{\partial u}{\partial \theta} = -r \sin \theta \frac{\partial u}{\partial x} + r \cos \theta \frac{\partial u}{\partial y}$$

converts partial derivatives in x and y to the sum in r and θ .

Now second derivatives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial r} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial r} \right)$$

$$= \cos \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right)$$

$$= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} + \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} - \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta}$$

$$\frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r} - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \sin \theta \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial r} \right) + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial r} \right)$$

$$= \sin \theta \frac{\partial}{\partial r} \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right) + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right)$$

$$= \sin^2 \theta \frac{\partial^2 u}{\partial r^2} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta}$$

$$\frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$