Laplace equation on a disk

PDE $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$ $\theta \in [-\pi, \pi]$, $r = [0, a]$

B.C. $u(a,\theta) = f(\theta)$ } not homogeneous.

$u(r,-\pi) = u(r,\pi)$

$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$

} homogeneous (in θ direction) (periodic boundary cond.)

Separation of variables

$$u(r,\theta) = \phi(\theta)\, G(r)$$

Superposition

$$u(r,\theta) = \sum_{n=0}^{\infty} \phi_n(\theta)\, G_n(r)$$

$\phi(\theta) = c_1 \cos\sqrt{\lambda}\,\theta + c_2 \sin\sqrt{\lambda}\,\theta$ with $\sqrt{\lambda} = n$ $n = 1, 2, \ldots$

$\phi(\theta) = \text{const}$ for $\lambda = 0$,

$$\begin{cases} G(r) = c_1 r^n & \text{for } n = 1, 2, \ldots \\ G(r) = \cancel{c_1 \log r} + c_2 & \text{for } \lambda = 0 \end{cases}$$

let $G_n(r) = r^n$

Thus

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos n\theta + B_n \sin n\theta\right) r^n$$

Now solve for A's and B's using orthogonality so that

$$u(a,\theta) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos n\theta + B_n \sin n\theta\right)a^n = f(\theta)$$

Take "dot" product of both sides with the different ϕ's.

$$\int_{-\pi}^{\pi}\left(A_0 + \sum_{n=1}^{\infty}\left(A_n \cancel{\cos n\theta} + B_n \cancel{\sin n\theta}\right)a^n\right)d\theta = \int_{-\pi}^{\pi} f(\theta)\,d\theta.$$

$$2\pi A_0 = \int_{-\pi}^{\pi} f(\theta)\,d\theta, \quad \text{so} \quad A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

Next with $\cos m\theta$ ...

$$\int_{-\pi}^{\pi} \cos m\theta\left(A_0 + \sum_{n=1}^{\infty}\left(A_n \cos n\theta + B_n \sin n\theta\right)a^n\right)d\theta = \int_{-\pi}^{\pi} \cos m\theta\, f(\theta)\,d\theta$$

$$\int_{-\pi}^{\pi} (\cos m\theta)(A_m \cos m\theta)\, a^m\, d\theta = \int_{-\pi}^{\pi} \cos m\theta\, f(\theta)\,d\theta$$

$$\frac{\cancel{2}\pi}{\cancel{2}} A_m a^m = \int_{-\pi}^{\pi} \cos m\theta\, f(\theta)\,d\theta$$

$$\boxed{A_m = \frac{1}{a^m \pi}\int_{-\pi}^{\pi} \cos m\theta\, f(\theta)\,d\theta,} \quad m > 0$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

from the book p76

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$(n \geq 1) \quad \boxed{A_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta}$$

$$B_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta.$$

also using same technique.

CHAPTER 3

Fourier Series

Given a function $f: [-L, L] \to \mathbb{R}$ it's Fourier series is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \quad \text{and} \quad \begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \end{cases}$$

Question: Does $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$?

Question: If not, how do they differ?

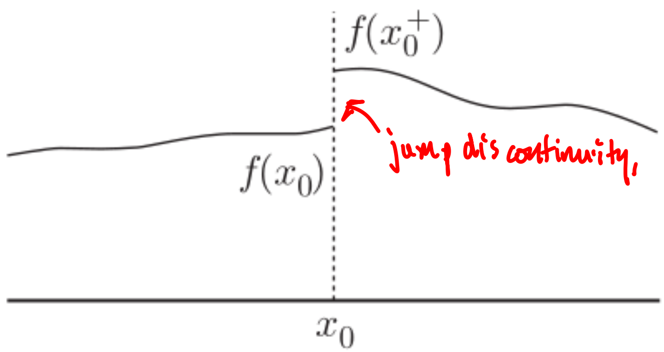
Question: Does the series even converge mathematically?
Do the integrals defining a's and b's exist?

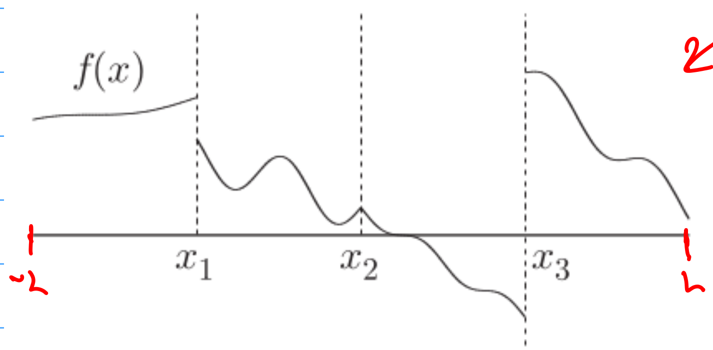
Convergence theorem:

If $f(x)$ is *piecewise smooth* on the interval $-L \leq x \leq L$, then the Fourier series of $f(x)$ converges

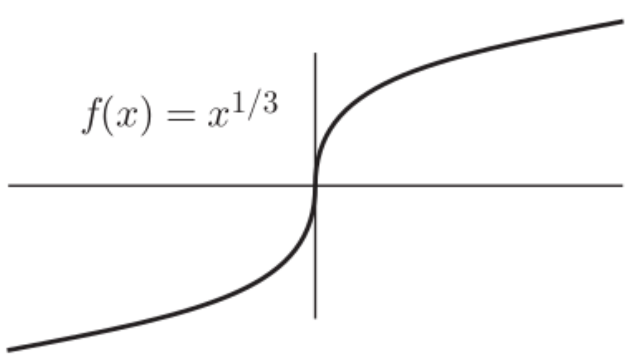
1. to the *periodic extension* of $f(x)$, where the periodic extension is continuous;
2. to the average of the two limits, usually
$$\frac{1}{2}\left[f(x+) + f(x-)\right],$$
where the periodic extension has a *jump discontinuity*.

easily, we will discuss only functions $f(x)$ that are piecewise smooth. A function $f(x)$ is **piecewise smooth** (on some interval) if the interval can be broken up into pieces (or sections) such that in each piece the function $f(x)$ is continuous[1] and its derivative df/dx is also continuous. The function $f(x)$ may not be continuous, but the only kind of discontinuity allowed is a finite number of jump discontinuities. A function $f(x)$ has **a jump discontinuity** at a point $x = x_0$ if the limit from the left $[f(x_0^-)]$ and the limit from the right $[f(x_0^+)]$ both exist (and are *unequal*), as illustrated in Fig. 3.1.1. An example of a piecewise smooth function is sketched in Fig. 3.1.2. Note that $f(x)$ has two jump discontinuities at $x = x_1$ and at $x = x_3$. Also, $f(x)$ is continuous for $x_1 \le x \le x_3$, but





piecewise smooth on [-L,L]

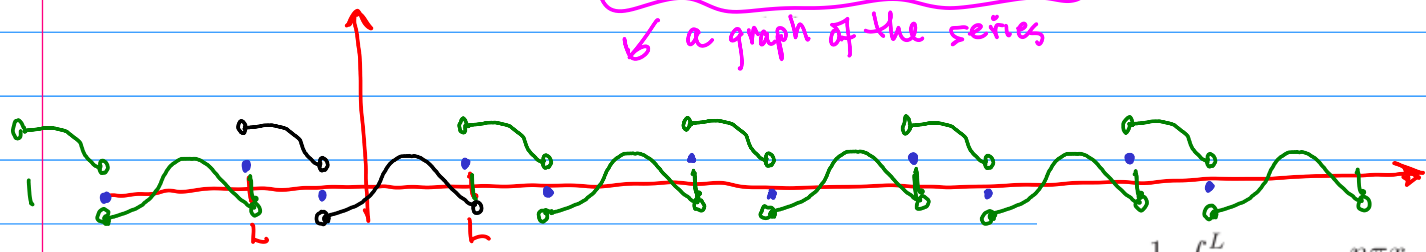


not piecewise smooth because the derivative does not exist at $x=0$.

If you try to break this up into pieces then $x=0$ has to be in one of the pieces and that's trouble...

Periodic extension: $a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$

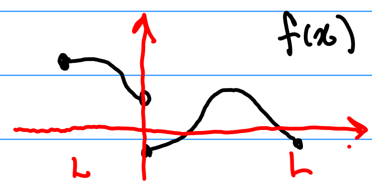
a graph of the series



$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

and

$$\begin{cases} a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \\ b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx. \end{cases}$$



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