Midterm is on Wednesday March 12

Idea: for each time t write the Fourier series of $u(t,x)$ as

$$u(t,x) \sim a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}$$

We did differentiation w.r.t. x last time

Convergence theorem for Fourier series. At first, we state a theorem summarizing certain properties of Fourier series:

> If $f(x)$ is *piecewise smooth* on the interval $-L \le x \le L$, then the Fourier series of $f(x)$ converges
> 1. to the *periodic extension* of $f(x)$, where the periodic extension is continuous;
> 2. to the average of the two limits, usually
> $$\frac{1}{2}[f(x+) + f(x-)],$$
> where the periodic extension has a *jump discontinuity*.

Last Friday

3.4 TERM-BY-TERM DIFFERENTIATION OF FOURIER SERIES

In solving partial differential equations by the method of separat

> A Fourier series that is continuous can be differentiated term by term if $f'(x)$ is piecewise smooth.

typo

> If $f'(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f(-L) = f(L)$.

Note. if the function f has a derivative that is piecewise smooth this means f is cont and also $f(-L) = f(L)$ further implies the periodic extension of f is continuous.

here is differentiating with respect to time.

The Fourier series of a continuous function $u(x,t)$ (depending on a parameter t)

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty}\left[a_n(t)\cos\frac{n\pi x}{L} + b_n(t)\sin\frac{n\pi x}{L}\right]$$

can be differentiated term by term with respect to the parameter t, yielding

$$\frac{\partial}{\partial t}u(x,t) \sim a_0'(t) + \sum_{n=1}^{\infty}\left[a_n'(t)\cos\frac{n\pi x}{L} + b_n'(t)\sin\frac{n\pi x}{L}\right]$$

if $\partial u/\partial t$ is piecewise smooth.

This is again interchanging the limiting operations of taking a derivative with the infinite sum...

After the exam we'll consider simpler case where the periodic extension of $\frac{\partial u}{\partial t}$ is smooth everywhere...

Now about the exam...

and also integrating factor for solving linear ODE...

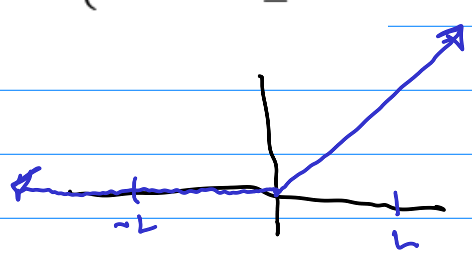
The midterm will be given in class on Wednesday, March 12. Please be prepared to find equilibrium solutions of the heat equation for different boundary conditions. You will also need to know how to solve time-dependent heat equations and the Laplace equation using separation of variables. Please review the quiz and suggested homework problems. There are partial solutions to the starred problems in the back of the book.

Here is a sample midterm to help you study.

The last Quiz on Fourier series could be extra credit on the midterm but not a regular question

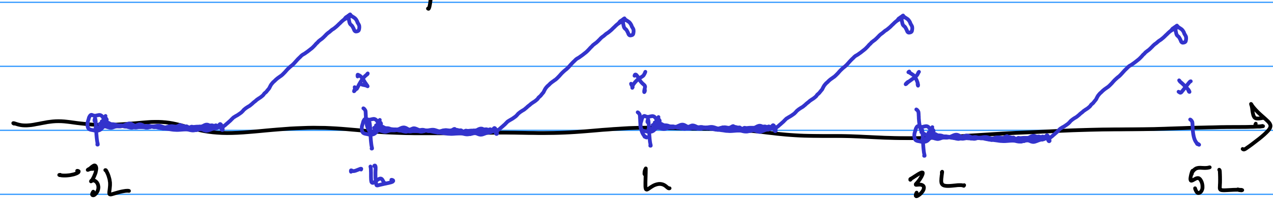
$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$$

graph of f

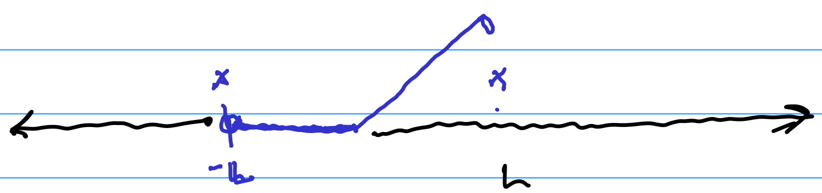


piecewise smooth so by the convergence theorem the Fourier series converges to the periodic extension where the function is continuous and the average of the jumps where it's not.

The Fourier series of f



The Fourier series of f on -L to L.



1. Recall the one-dimensional heat equation with constant thermal properties given by

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L].$$

Here c is the heat capacity, ρ the density, K_0 the conductivity, Q the rate of production of heat energy and u the temperature. Suppose $L = 1$ and $Q/K_0 = x^2 - x$. If the initial condition and boundary conditions satisfy

$$u(x,0) = \cos(\pi x/2) \quad \text{for} \quad x \in [0,1]$$
$$u(0,t) = 1 \quad \text{and} \quad u(1,t) = 0 \quad \text{for} \quad t > 0,$$

find the equilibrium temperature of the rod obtained as $t \to \infty$.

The ODE for the equilibrium is $u''(x) = -\frac{Q}{K_0}$

$$u''(x) = -x^2 + x$$
$$u'(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$
$$u(x) = -\frac{1}{12}x^4 + \frac{1}{6}x^3 + Cx + D$$

B.C. $u(0)=1$ and $u(1)=0$

$$u(0) = -\frac{1}{12}0^4 + \frac{1}{6}0^3 + C0 + D = 1 \quad \text{so } D=1$$

$$u(1) = -\frac{1}{12}1^4 + \frac{1}{6}1^3 + C\cdot 1 + 1 = 0$$

$$C = \frac{1}{12} - \frac{1}{6} - 1 = -\frac{1}{12} - 1 = -\frac{13}{12}$$

Solution $u(x) = -\frac{1}{12}x^4 + \frac{1}{6}x^3 - \frac{13}{12}x + 1$

2. For the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

what ordinary differential equations are implied by the method of separation of variables? Do not solve to ODEs.

plug in $u(r,t) = \varphi(r)\,G(t)$

$$\varphi(r)\,G'(t) = \frac{k}{r}G(t)\frac{d}{dr}\left(r\varphi'(r)\right)$$

Separate variables

$$\underbrace{\frac{G'(t)}{k\,G(t)}}_{\text{no } r \text{ dependence}} = \underbrace{\frac{1}{r\varphi(r)}\frac{d}{dr}\left(r\varphi'(r)\right)}_{\text{no } t \text{ dependence}} = -\lambda$$

const does not depend on r or t.

ODES

$$\boxed{\begin{aligned} &\bullet\ G'(t) = -\lambda k\,G(t) \\ &\bullet\ \frac{d}{dr}\left(r\varphi'(r)\right) = -\lambda r\varphi(r) \end{aligned}}$$

$$r\varphi''(r) + \varphi'(r) + \lambda r\varphi(r) = 0$$

optional to expand it like this ... this is a Bessel equation but not part of the question

3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L]$$

subject to the homogeneous boundary conditions

heat bath insulating

$$u(0,t) = 0 \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

Solve the initial value problem if the temperature is initially

$$u(x,0) = 5\sin\left(\frac{3\pi x}{2L}\right).$$

$$u(x,t) = \varphi(x)\,G(t)$$

$$\varphi(x)\,G'(t) = k\,G(t)\,\varphi''(x)$$

$$\frac{G'(t)}{k\,G(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

ODEs $\varphi''(x) = -\lambda\varphi(x)$ and $G'(t) = -k\lambda G(t)$

$\varphi(0) = 0$

$\varphi'(L) = 1$

use superposition to solve for boundary.

Gen soln

$\lambda > 0$

$\varphi(x) = c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x$ $\qquad G(t) = e^{-\lambda k t}$

ran out time ... Work continued after class ...

$\varphi(0) = c_1 = 0$ implies $c_1 = 0$

$\varphi'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x$

$\varphi'(L) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,L = 0$

since $c_2 \neq 0$ then $\cos\sqrt{\lambda}\,L = 0$ so $\sqrt{\lambda}\,L = \frac{\pi}{2} + n\pi = \frac{(2n+1)\pi}{2}$

so $\sqrt{\lambda} = \frac{(2n+1)\pi}{2L}$ $\quad n = 1, 2, \ldots$

$\varphi_n(x) = \sin\frac{(2n+1)\pi x}{2L}$ for $n = 1, 2, \ldots$

$\lambda = 0$

$\varphi(x) = C_1 x + C_2$

$\varphi(0) = C_2 = 0$ so $C_2 = 0$

$\varphi'(x) = C_1 = 0$ so $C_1 = 0$

no non-zero solutions

$\lambda < 0$

also no non-zero solutions

Solve the other ODE $\lambda = \left(\frac{(2n+1)\pi}{2L}\right)^2$

$$G(t) = e^{-k\left(\frac{(2n+1)\pi}{2L}\right)^2 t}$$

The superposition looks like

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{(2n+1)\pi x}{2L} \, e^{-k\left(\frac{(2n+1)\pi}{2L}\right)^2 t}$$

to satisfy the initial condition

$$u(x,0) = 5 \sin \frac{3\pi x}{2L}$$

Since $2n+1 = 3$ when $n=1$ then $a_1 = 5$ and $a_n = 0$ for $n \neq 1$.

Consequently,

$$\boxed{u(x,t) = 5 \sin \frac{3\pi x}{2L} \, e^{-k\left(\frac{3\pi}{2L}\right)^2 t}}$$