The Fourier series of a continuous function $u(x,t)$ (depending on a parameter t)

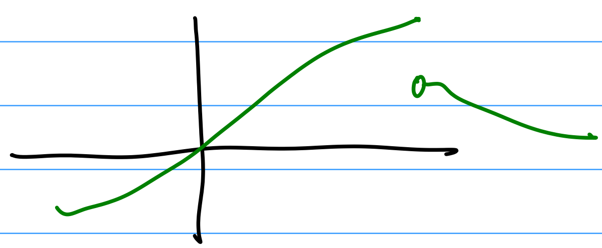
$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty}\left[\underbrace{a_n(t)}_{\alpha_n(t)}\cos\frac{n\pi x}{L} + \underbrace{b_n(t)}_{\beta_n(t)}\sin\frac{n\pi x}{L}\right]$$

can be differentiated term by term with respect to the parameter t, yielding

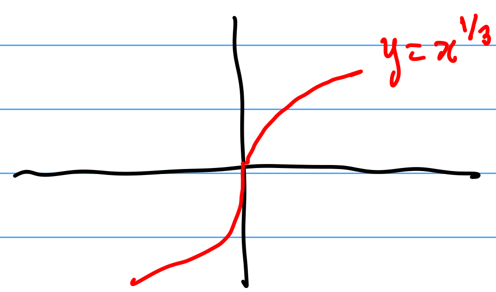
$$\frac{\partial}{\partial t}u(x,t) \sim a_0'(t) + \sum_{n=1}^{\infty}\left[a_n'(t)\cos\frac{n\pi x}{L} + b_n'(t)\sin\frac{n\pi x}{L}\right]$$

if $\underbrace{\partial u/\partial t}_{u_t}$ is piecewise smooth.

Since u_t is piecewise smooth, then on each piece u_{tt} exists and is continuous.



piecewise smooth



not piecewise smooth.

Claim that u_{tt} is bounded everywhere ...

(Assume).

by Fourier convergence theorem this converges to u_t where u_t is cont and to the avg. of the jump discontinuities...

fourier series of the derivative

$$u_t(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty}\left(\alpha_n(t)\cos\frac{n\pi x}{L} + \beta_n(t)\sin\frac{n\pi x}{L}\right)$$

where $\alpha_0(t) = \frac{1}{2L}\int_{-L}^{L} u_t(x,t)\,dx \qquad \alpha_n(t) = \frac{1}{L}\int_{-L}^{L} u_t(x,t)\cos\frac{n\pi x}{L}\,dx$

$$\beta_n(t) = \frac{1}{L}\int_{-L}^{L} u_t(x,t)\sin\frac{n\pi x}{L}\,dx$$

Want to compare to

term by term derivative of the Fourier series of u.

$$a_0'(t) + \sum_{n=1}^{\infty}\left[a_n'(t)\cos\frac{n\pi x}{L} + b_n'(t)\sin\frac{n\pi x}{L}\right]$$

Equality means $\underbrace{a_0'(t) = \alpha_0(t),\ a_n'(t) = \alpha_n(t),\ b_n'(t) = \beta_n(t)}_{\text{what I need to show.}}$

$$a_0(t) = \frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx \qquad a_n(t) = \frac{1}{L}\int_{-L}^{L} u(x,t)\cos\frac{n\pi x}{L}\,dx$$

$$b_n(t) = \frac{1}{L}\int_{-L}^{L} u(x,t)\sin\frac{n\pi x}{L}\,dx$$

$$a_0'(t) = \frac{d}{dt}\left(\frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx\right)$$

like to switch the derivative and integral..

$$a_n(t) = a_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

Recall from 2025-02-12 Lecture 10

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}}$$

where $a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

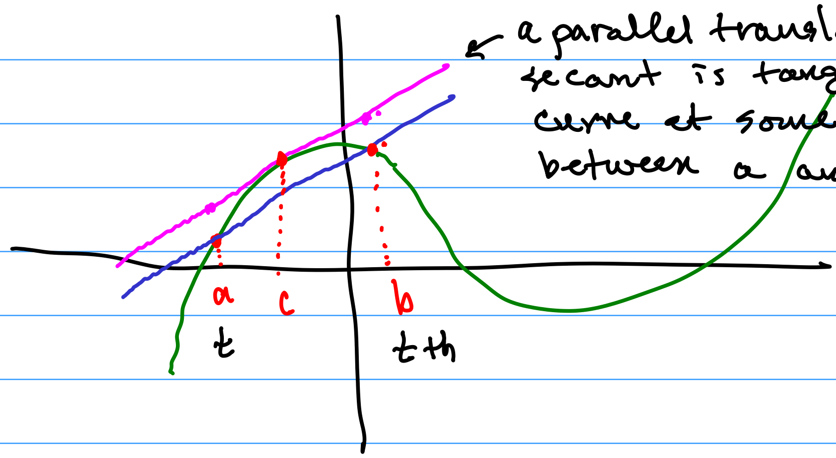
$$a_0'(t) = \frac{d}{dt}\left(\frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx\right) = \lim_{h\to 0}\frac{\frac{1}{2L}\int_{-L}^{L} u(x,t+h)\,dx - \frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx}{h}$$

$$= \lim_{h\to 0}\frac{1}{2hL}\int_{-L}^{L}\big(u(x,t+h)-u(x,t)\big)\,dx$$

trying to get derivatives inside the integral.

One idea mean value theorem.

Mean value theorem



a parallel translation of the secant is tangent to the curve at some point between a and b.

$$\frac{u(x,t+h)-u(x,t)}{h} = u_t(x,c)$$

here c depends on everything

$c = c(x,t,h)$

Iterate mean value theorem, we get Taylor's theorem

$$u(x,t+h) = u(x,t) + h\,u_t(x,c)$$

$$u(x,t+h) = u(x,t) + h\,u_t(x,t) + \frac{h^2}{2}u_{tt}(x,\xi)$$

where $\xi = \xi(x,t,h)$

apply on each piece since u_t is piecewise smooth

Use Taylor's theorem to put derivatives inside the integral.

plug in ..

$$\lim_{h\to 0} \frac{1}{2hL}\int_{-L}^{L} \left(u(x,t+h) - u(x,t)\right) dx$$

$$= \lim_{h\to 0} \frac{1}{2hL}\int_{-L}^{L} \left(h\, u_t(x,t) + \frac{h^2}{2} u_{tt}(x,\xi)\right) dx$$

$$= \underbrace{\frac{1}{2L}\int_{-L}^{L} u_t(x,t)\,dx}_{\alpha_0(t)} + \lim_{h\to 0} \frac{h}{4L}\int_{-L}^{L} u_{tt}(x,\xi(x,t,h))\,dx$$

by hypothesis $|u_{tt}| < B$ for some bound.

$$\lim_{h\to 0}\left|\frac{h}{4L}\int_{-L}^{L} u_{tt}(x,\xi(x,t,h))\,dx\right| \le \lim_{h\to 0}\frac{h}{4L}\int_{-L}^{L} \left|u_{tt}(x,\xi(x,t,h))\right| dx .$$

$$\lim_{h\to 0}\frac{h}{4L}\int_{-L}^{L} B\,dx = \lim_{h\to 0} \frac{h}{2} B = 0$$

Since:

$$\alpha_0(t) = \frac{1}{2L}\int_{-L}^{L} u_t(x,t)\,dx$$

$$\lim_{h\to 0} \frac{1}{2hL}\int_{-L}^{L} \left(u(x,t+h) - u(x,t)\right) dx = \alpha_0(t)$$

so $$\frac{d}{dt} a_0(t) = \alpha_0(t).$$