$$u_t(x,t) = \sum_{n=1}^{\infty}\left(-A_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L} \sin\frac{n\pi c t}{L} + B_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L} + B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi c t}{L}\right)$$

Use the initial conditions and orthogonality to solve for A_n and B_n

$$\text{IC:} \quad \boxed{\begin{aligned} &u(x,0) = f(x) \\ &\frac{\partial u}{\partial t}(x,0) = g(x), \end{aligned}}$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x) \qquad u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L} = g(x)$$

$$\int_0^L \sum_{n=1}^{\infty} A_n \underbrace{\sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}}_{\text{orthogonality here}} dx = \int_0^L f(x) \sin\frac{m\pi x}{L}\, dx$$

what is this?

trigonometry $\quad \sin(a+b) = \sin a \cos b + \cos a \sin b$

$$\cos(a+b) = \frac{\partial}{\partial a}\sin(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\cos(a+b) - \cos(a-b) = -2 \sin a \sin b$$

$$\int_0^L \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L}\, dx = \int_0^L -\frac{1}{2}\left(\cos\frac{(n+m)\pi x}{L} - \cos\frac{(n-m)\pi x}{L}\right) dx$$

note both m, n are positive integers.

Case $m=n$

$$\int_0^L -\frac{1}{2}\left(\cos\frac{(n+m)\pi x}{L} - \cos\frac{(n-m)\pi x}{L}\right)dx = \int_0^L -\frac{1}{2}\left(\cos\frac{2m\pi x}{L} - 1\right)dx$$

$$= -\frac{1}{2}\left(\frac{L}{2m\pi}\sin\frac{2m\pi x}{L} - x\right)\Bigg|_0^L$$

$$= -\frac{1}{2}\left(\frac{L}{2m\pi}\sin\underbrace{\frac{2m\pi L}{L}}_{\text{also } 0} - L\right) + \frac{1}{2}\left(\frac{L}{2m\pi}\sin\frac{2m\pi 0}{L} - 0\right) = \frac{L}{2}$$

Case $m \neq n$ then $m-n \neq 0$ and $m+n \neq 0$ (because m, n are positive)

$$\int_0^L -\frac{1}{2}\left(\cos\frac{(n+m)\pi x}{L} - \cos\frac{(n-m)\pi x}{L}\right)dx$$

$$= -\frac{1}{2}\left(\frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L} - \frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L}\right)\Bigg|_0^L = 0$$

Therefore,

$$\int_0^L \sin\frac{n\pi x}{L}\cdot\sin\frac{m\pi x}{L}\,dx = \begin{cases} \frac{L}{2} & \text{if} \quad m=n \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

interchange the sum and integral ...

$$\int_0^L \sum_{n=1}^{\infty} \underbrace{A_n \sin\frac{n\pi x}{L}\ \sin\frac{m\pi x}{L}}_{\text{orthogonality here}}\,dx = \int_0^L f(x)\ \sin\frac{m\pi x}{L}\,dx$$

$$\sum_{n=1}^{\infty} A_n \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \int_0^L f(x) \sin\frac{m\pi x}{L}\, dx$$

Therefore

$$A_m \frac{L}{2} = \int_0^L f(x) \sin\frac{m\pi x}{L}\, dx$$

$$\boxed{A_m = \frac{2}{L}\int_0^L f(x) \sin\frac{m\pi x}{L}\, dx}$$

The other one is

$$\sum_{n=1}^{\infty} \underbrace{B_n \frac{n\pi c}{L}}_{\text{plays the role of } A_n} \sin\frac{n\pi x}{L} = \underbrace{g(x)}_{\text{role of } f}$$

$$\boxed{B_m \frac{m\pi c}{L} = \frac{2}{L}\int_0^L g(x) \sin\frac{m\pi x}{L}\, dx}$$

Therefore

$$B_m = \frac{2}{\cancel{L}} \frac{\cancel{L}}{m\pi c} \int_0^L g(x) \sin\frac{m\pi x}{L}\, dx = \frac{2}{m\pi c}\int_0^L g(x) \sin\frac{m\pi x}{L}\, dx$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi c t}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi c t}{L}\right)$$

trigonometry

$$\sin(a-b) = \sin a\cos b - \cos a \sin b$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \frac{\partial}{\partial a}\sin(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\cos(a+b) - \cos(a-b) = -2\sin a \sin b$$
$$\sin(a+b) + \sin(a-b) = 2\sin a \cos b$$

$$\sin\frac{n\pi x}{L}\sin\frac{n\pi c t}{L} = -\frac{1}{2}\left(\cos\frac{n\pi(x+ct)}{L} - \cos\frac{n\pi(x-ct)}{L}\right)$$

$$\sin\frac{n\pi x}{L}\cos\frac{n\pi c t}{L} = \frac{1}{2}\left(\sin\frac{n\pi(x+ct)}{L} + \sin\frac{n\pi(x-ct)}{L}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n\left(\frac{1}{2}\right)\left(\sin\frac{n\pi(x+ct)}{L} + \sin\frac{n\pi(x-ct)}{L}\right)$$
$$+ \sum_{n=1}^{\infty} B_n\left(-\frac{1}{2}\right)\left(\cos\frac{n\pi(x+ct)}{L} - \cos\frac{n\pi(x-ct)}{L}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{A_n}{2}\sin\frac{n\pi(x+ct)}{L} - \frac{B_n}{2}\cos\frac{n\pi(x+ct)}{L}\right)$$
$$+ \sum_{n=1}^{\infty}\left(\frac{A_n}{2}\sin\frac{n\pi(x-ct)}{L} + \frac{B_n}{2}\cos\frac{n\pi(x-ct)}{L}\right)$$

$$= S(x+ct) + R(x-ct)$$

where

the solution is a sum of two traveling waves one going left and the other right.

$$S(w) = \sum_{n=1}^{\infty} \left(\frac{A_n}{2} \sin \frac{n\pi w}{L} - \frac{B_n}{2} \cos \frac{n\pi w}{L} \right)$$

$$R(w) = \sum_{n=1}^{\infty} \left(\frac{A_n}{2} \sin \frac{n\pi w}{L} + \frac{B_n}{2} \cos \frac{n\pi w}{L} \right)$$

S left	R right	7 $\boxed{\checkmark}$
S right	R left	2

$S(x+ct) + R(x-ct)$