this term is non-linear.

$$1\frac{\partial\rho}{\partial t} + \boxed{\rho\frac{\partial\rho}{\partial x}} = t \qquad \rho(x,0) = f(x)$$

Characteristics write $x = x(s)$ and $t = t(s)$

$$x'(s) = \rho(x(s), t(s)) = \rho(s)$$

$$t'(s) = 1$$

$t = s + c_1$

if $s=0$ then $t=0$

thus $c_1 = 0$

and $t = s$

If $x(s)$ and $t(s)$ satisfy the ODEs then

$$\frac{d\rho}{ds} = t(s)$$

This suggest a shortcut when we see the coefficient of $\frac{\partial\rho}{\partial t}$ is 1 and the initial condition is like $\rho(x,0) = f(x)$ then we could just start with

Characteristics write $x = x(t)$.

$$\underbrace{x'(t) = \rho(t)}_{\text{can't find the characteristic unless I know } \rho.} \qquad \text{and} \qquad \underbrace{\frac{d}{dt}\rho(x(t), t) = t}_{\text{solve for } \rho \text{ first}}$$

$$\rho(x(t), t) = \frac{1}{2}t^2 + c_2$$

Once I know what ρ would be if on the characteristic then I can solve for the characteristic.

$$\rho(x(t), t) = \frac{1}{2}t^2 + c_2 \qquad \rho(x,0) = f(x)$$

solve for c_2 by setting $t=0$

$$\rho(x(0), 0) = \frac{1}{2}0^2 + c_2 = f(x(0))$$

thus $c_2 = f(x(0)) = f(x_0)$

writing $x_0 = x(0)$.

$$\rho(x(t),t) = \frac{1}{2}t^2 + f(x_0)$$

Now that we know ρ then find x.

$$x'(t) = \frac{1}{2}t^2 + f(x_0)$$

Substitute

$$x(t) = \frac{1}{6}t^3 + f(x_0)t + C_3$$

$$x(0) = \frac{1}{6}0^3 + f(x_0)\cdot 0 + C_3 = x_0 \qquad C_3 = x_0$$

Solution

implicit form of the solution

$$\boxed{\rho\left(\frac{1}{6}t^3 + f(x_0)t + x_0,\ t\right) = \frac{1}{2}t^2 + f(x_0)}$$

Solve for $\rho(x,t)$. Set $x = \frac{1}{6}t^3 + f(x_0)t + x_0$

Now solve for x_0 in terms of x...

We need to know what f is to do that

Implicit solution.

$$\rho(x,t) = \frac{1}{2}t^2 + f(x_0) \quad \text{where} \quad x = \frac{1}{6}t^3 + f(x_0)t + x_0$$

Solving for x_0 may or may not be possible and depends of f. and the implicit form of the solution happened because of the non-linear term

$$1\frac{\partial \rho}{\partial t} + \boxed{\rho\frac{\partial \rho}{\partial x}} = t$$

Example: $f(x) = 7$

$$x = \frac{1}{6}t^3 + 7t + x_0$$

$$x_0 = x - \frac{1}{6}t^3 - 7t$$

could skip solving for x_0 because $f(x_0)$ didn't depend on x_0.

$$\rho(x,t) = \frac{1}{2}t^2 + f(x_0) = \frac{1}{2}t^2 + f\left(x - \frac{1}{6}t^3 - 7t\right)$$

$$\rho(x,t) = \frac{1}{2}t^2 + 7$$

Example : $f(x) = ax + b$.

$$x = \frac{1}{6}t^3 + (ax_0 + b)t + x_0 = \frac{1}{6}t^3 + ax_0t + bt + x_0$$

$$= \frac{1}{6}t^3 + bt + x_0(at+1)$$

$$x_0 = \frac{x - \frac{1}{6}t^3 - bt}{at+1}$$

The solution is

$$\rho(x,t) = \frac{1}{2}t^2 + f\left(\frac{x - \frac{1}{6}t^3 - bt}{at+1}\right) = \frac{1}{2}t^2 + a\left(\frac{x - \frac{1}{6}t^3 - bt}{at+1}\right) + b$$

$$\rho(x,t) = \frac{1}{2}t^2 + a\left(\frac{x - \frac{1}{6}t^3 - bt}{at+1}\right) + b$$

what if $at = -1$?

this is guaranteed to happen if $a < 0$ for some $t > 0$

If $a > 0$ then $at + 1 \neq 0$ for all $t > 0$.

If $a < 0$ then there is a shock in the solution at $t = \frac{-1}{a}$.

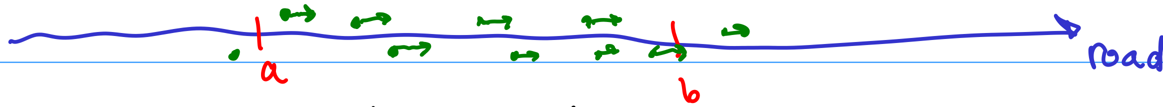
Try to understand shock behavior from a physical example.

Traffic density and flow. As an approximation it is possible to model a congested one-directional highway by a quasilinear partial differential equation. We introduce the **traffic density** $\rho(x, t)$, the number of cars per mile at time t located at position x. An easily observed and measured quantity is the **traffic flow** $q(x, t)$, the number of cars per hour passing a fixed place x (at time t).

$\rho(x,t)$ = cars/mile at point x at time t (density)

$q(x,t)$ = cars/hour passing a fixed place x at time t

(traveling left to right)



How many cars are on the strech of road from a to b.

$$\int_a^b \rho(x,t)\,dx$$

$$\underbrace{\frac{d}{dt}\int_a^b \rho(x,t)\,dx}_{\text{rate \# of cars change over time}} = \underbrace{q(a,t)}_{\text{rate they enter}} - \underbrace{q(b,t)}_{\text{rate they leave.}} = -\int_a^b \frac{\partial}{\partial x} q(x,t)\,dx$$

Thus

$$\int_a^b \left[\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} q(x,t)\right] dx = 0$$

Since the above holds for any endpoints a and b, then the only way the above integral is always 0 is if

PDE $$\frac{\partial}{\partial t}\rho(x,t) + \underbrace{\frac{\partial}{\partial x} q(x,t)}_{\text{with respect to } x} = 0 .$$

We need something like Fourier's law to connect ρ and q.

Idea: Cars drive faster when there are fewer cars nearby on the road.

Okay: There are speed limits and other things...

Suppose the speed $u(\rho(x,t))$ = miles/hour depends of the density of cars at point x at time t.

$$q(x,t) = u(\rho(x,t))\,\rho(x,t)$$

$$\frac{\text{cars}}{\text{hour}} = \frac{\text{miles}}{\text{hour}} \; \frac{\text{cars}}{\text{mile}}$$

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = q(a,t) - q(b,t) = \underbrace{u(\rho)\,\rho}_{\varphi(\rho)}\Big|_{\rho=\rho(b,t)}^{\rho(a,t)}$$

Let $\varphi(\rho) = u(\rho)\rho$

$$\frac{\partial}{\partial x} q(x,t) = \frac{\partial}{\partial x}\varphi(\rho(x,t)) = \varphi'(\rho(x,t))\,\frac{\partial \rho}{\partial x}$$

substitute back in... next time...