Suppose the speed $u(\rho(x,t))$ = miles/hour depends of the density of cars at point x at time t,

$$\psi(\rho) = q(x,t) = \boxed{u(\rho(x,t))\,\rho(x,t)}$$

$$\frac{\text{cars}}{\text{hour}} = \frac{\text{miles}}{\text{hours}} \quad \frac{\text{cars}}{\text{mile}}$$

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = q(a,t) - q(b,t) = \underbrace{u(\rho)\rho}_{\psi(\rho)}\Big|_{\rho=\rho(b,t)}^{\rho(a,t)}$$

Let $\psi(\rho) = u(\rho)\rho$

$$\frac{\partial}{\partial x} q(x,t) = \frac{\partial}{\partial x}\psi(\rho(x,t)) = \psi'(\rho(x,t))\frac{\partial \rho}{\partial x}$$

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} q(x,t) = 0 .$$

with respect to x

Substitute

$$\frac{\partial \rho}{\partial t} + \underbrace{\psi'(\rho)}_{c(\rho)}\frac{\partial \rho}{\partial x} = 0$$

The distribution of cars at $t=0$

Then

$$1\,\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \qquad \rho(x,0) = f(x)$$

Solve by the method of characteristics

Note the coefficient on $\frac{\partial \rho}{\partial t}$ is 1 and the initial cond, is at $t=0$.

Therefore instead of parameterizing the characteristics by s and the substituting $s=t$, Just start with t as the parameter.

$$x = x(t) \qquad t = t.$$

$$\rho(t) = \rho(x(t), t)$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x} x'(t) + \frac{\partial \rho}{\partial t}$$

$$\rho(x,0) = f(x)$$

The ODEs are

$$x'(t) = c(\rho) \qquad \frac{d\rho}{dt} = 0$$

so $\rho(x(t), t) = c_1$

$\rho(\underbrace{x(0)}, 0) = c_1 = f(x(0))$

$x_0 = x(0)$ Notation

thus $\rho(x(t), t) = f(x_0)$

$$x'(t) = c(f(x_0))$$

$$x(t) = c(f(x_0))t + c_2 \qquad x(0) = c_2 = x_0$$

Therefore

$$\rho(c(f(x_0))t + x_0, t) = f(x_0)$$

$\rho(x,t) = f(x_0)$ where $x = c(f(x_0))t + x_0$

Talk about this solution more on friday.