

$$\text{Fourier series } g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx.$$

Converges pointwise provided.

$f$  is piecewise smooth.

$$g'(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} \beta_n \sin \frac{n\pi x}{L}$$

$$\alpha_0 = \frac{1}{2L} \int_{-L}^L f'(x) dx$$

$$\alpha_n = \frac{1}{L} \int_{-L}^L f'(x) \cos \frac{n\pi x}{L} dx$$

$$\beta_n = \frac{1}{L} \int_{-L}^L f'(x) \sin \frac{n\pi x}{L} dx$$

Converges pointwise provided  $g'$  is

piecewise smooth...

Note:  $g'$  is the periodic extension of  $f'$  with  $x$ 's added at the jump discontinuities at the average value of the jump.

$$\frac{d}{dx} g(x) = \frac{d}{dx} \left( a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \right)$$

$$\beta_n = -\frac{n\pi}{L} a_n$$

If I could differentiate term by term then

$$g'(x) \stackrel{?}{=} \underbrace{0}_{\alpha_0} + \sum_{n=1}^{\infty} \underbrace{-\frac{n\pi}{L} a_n}_{\beta_n} \sin \frac{n\pi x}{L} + \sum_{n=1}^{\infty} \underbrace{\frac{n\pi}{L} b_n}_{\alpha_n?} \cos \frac{n\pi x}{L}$$

If  $f'(x)$  is piecewise smooth, then the Fourier series of a continuous function  $f(x)$  can be differentiated term by term if  $f(-L) = f(L)$ .

$$a_0 = \frac{1}{2L} \int_{-L}^L f'(x) dx = \frac{1}{2L} f(x) \Big|_{-L}^L = \frac{1}{2L} (f(L) - f(-L)) = 0$$

since  $f(-L) = f(L)$   
by hypothesis

need to get the prime  
away from the f. integration by parts...

$$a_n = \frac{1}{L} \int_{-L}^L f'(x) \cos \frac{n\pi x}{L} dx$$

$$u = \cos \frac{n\pi x}{L}$$

$$dv = f'(x) dx$$

$$du = -\frac{n\pi}{L} \sin \frac{n\pi x}{L}$$

$$v = f(x)$$

$$= \frac{1}{L} \left( \cos \frac{n\pi x}{L} f(x) \Big|_{-L}^L + \int_{-L}^L \frac{n\pi}{L} \sin \frac{n\pi x}{L} f(x) dx \right)$$

$$= \frac{1}{L} \left[ (\cos n\pi) f(L) - \cos(-n\pi) f(-L) \right] + \frac{1}{L} \int_{-L}^L \frac{n\pi}{L} \sin \frac{n\pi x}{L} f(x) dx$$

$$= \frac{n\pi}{L} \cdot \frac{1}{L} \int_{-L}^L \sin \frac{n\pi x}{L} f(x) dx$$

$$a_n = \frac{n\pi}{L} b_n \quad \checkmark$$

recall

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx.$$

$$\beta_n = \frac{1}{L} \int_{-L}^L f'(x) \sin \frac{n\pi x}{L} dx$$

$$u = \sin \frac{n\pi x}{L}$$

$$dv = f'(x) dx$$

$$du = \frac{n\pi}{L} \cos \frac{n\pi x}{L}$$

$$v = f(x)$$

$$= \frac{1}{L} \left( \sin \frac{n\pi x}{L} f(x) \Big|_{-L}^L - \int_{-L}^L \frac{n\pi}{L} \cos \frac{n\pi x}{L} f(x) dx \right)$$

$$= \frac{1}{L} \left[ \left( \sin n\pi \right) f(L) - \sin(-n\pi) f(-L) \right] - \frac{1}{L} \int_{-L}^L \frac{n\pi}{L} \cos \frac{n\pi x}{L} f(x) dx$$

here the fact that  $\sin(n\pi) = 0$  is used.

$$= \frac{n\pi}{L} - \frac{1}{L} \int_{-L}^L \cos \frac{n\pi x}{L} f(x) dx = -\frac{n\pi}{L} a_n$$

Thus

$$\beta_n = -\frac{n\pi}{L} a_n$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

Thus the differentiation of the Fourier series with respect to  $x$  depends on integration by parts and we need to check the hypothesis of the integration by parts formula to really have a proof.

Recall  $uv$  apply product rule

$$(uv)' = u'v + uv'$$

$$\int (uv)' dx = \int u'v dx + \int uv' dx$$

fund theorem  
of calculus

$$uv = \int u'v dx + \int uv' dx$$

$$\int_a^b uv' dx = uv \Big|_a^b - \int_a^b u'v dx$$

In our case we are working with

$$\int_{-L}^L f'(x) \sin \frac{n\pi x}{L} dx$$

this function is very smooth

only problems here...

In this case integration by parts works even when  $f'$  is only piecewise smooth...

#### Validity for less smooth functions [\[edit\]](#)

It is not necessary for  $u$  and  $v$  to be continuously differentiable. Integration by parts works if  $u$  is [absolutely continuous](#) and the function designated  $v'$  is [Lebesgue integrable](#) (but not necessarily continuous).<sup>[3]</sup> (If  $v'$  has a point of discontinuity then its antiderivative  $v$  may not have a derivative at that point.)

If the interval of integration is not [compact](#), then it is not necessary for  $u$  to be absolutely continuous in the whole interval or for  $v'$  to be Lebesgue integrable in the interval, as a couple of examples (in which  $u$  and  $v$  are continuous and continuously differentiable) will show. For instance, if

since sine and cosine are so nice  
the formula works for us...