Math 488: Sample Midterm Version A

This is a closed-book closed-notes no-calculator-allowed in-class exam. Efforts have been made to keep the arithmetic simple. If it turns out to be complicated, that's either because I made a mistake or you did. In either case, do the best you can and check your work where possible. While getting the right answer is nice, this is not an arithmetic test. It's more important to clearly explain what you did and what you know.

1. Recall the one-dimensional heat equation with constant thermal properties given by

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L].$$

Here c is the heat capacity, ρ the density, K_0 the conductivity, Q the rate of production of heat energy and u the temperature. Suppose $L = 1$ and $Q/K_0 = x^2 - x$. If the initial condition and boundary conditions satisfy

$$u(x, 0) = \cos(\pi x/2) \quad \text{for} \quad x \in [0, 1]$$
$$u(0, t) = 1 \quad \text{and} \quad u(1, t) = 0 \quad \text{for} \quad t > 0,$$

find the equilibrium temperature of the rod obtained as $t \to \infty$.

2. For the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

what ordinary differential equations are implied by the method of separation of variables?

3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L]$$

subject to the homogeneous boundary conditions

$$u(0, t) = 0 \qquad \text{and} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

Solve the initial value problem if the temperature is initially

$$u(x, 0) = 5\sin\left(\frac{3\pi x}{2L}\right).$$