Steady state solutions when $\frac{\partial u}{\partial t} = 0$. We think of these as what happens in the future as $t \to \infty$.

$$\cancel{\frac{\partial u}{\partial t}} = k \frac{\partial^2 u}{\partial x^2}$$

Thus $u(x,t) = u(x)$ and so we obtain the ODE.

$$\cancel{k} \frac{d^2 u}{dx^2} = 0$$

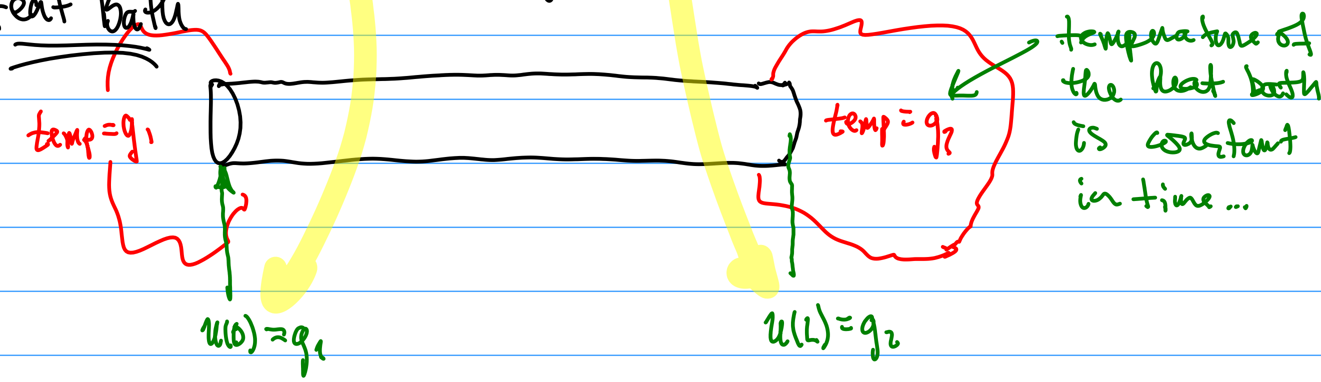
or $\frac{d^2 u}{dx^2} = 0$. What's the solution in general?

$$\frac{du}{dx} = \int 0 \, dx = C$$

$$\boxed{u(x) = \int C \, dx = Cx + D}$$

What about the boundary conditions?

Heat Bath



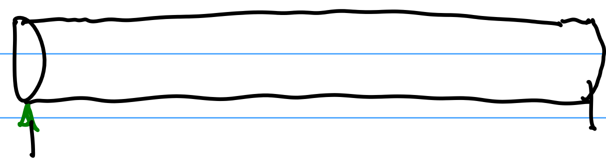
Substitute and solve for C and D.

$$u(0) = Cx + D\Big|_{x=0} \Rightarrow D = g_1 \qquad \text{so} \qquad D = g_1$$

$$u(L) = Cx\Big|_{x=L} = CL + g_1 = g_2 \qquad C = (g_2 - g_1)/L$$

Solution is $\boxed{u(x) = \frac{g_1 - g_2}{L}x + g_1}$

<u>Insulating Boundary</u>: $u(0,x) = f(x)$ initial distribution of heat



$q(0) = 0$ $\qquad\qquad$ $q(L) = 0$

$-K_0 \frac{du}{dx}\Big|_{x=0} = 0$ $\qquad\qquad$ $-K_0 \frac{du}{dx}\Big|_{x=L} = 0$

not independent boundary conditions. So can't solve for both constants...

$u = Cx + D$

$\frac{du}{dx}\Big|_{x=0} = C = 0$ $\qquad$ plug in C $\qquad$ $C = 0$

or $x = L$ same thing...

$\boxed{u = D}$ how to solve for D?

Since flux is zero energy in the rod is conserved...

$$\text{Energy} = \int_{\Omega} e \, dV = A \int_0^L e(s) \, ds$$

$$E = A \int_0^L c\rho \, u(s) \, ds = Ac\rho \int_0^L u(s) \, ds$$

energy density.

$$e(x) = c(x)\rho(x)u(x)$$

Energy at the start is the same as after $t \to \infty$ since the flux at the boundary is zero and no heat is produced in the rod.

$$E = Ac\rho \int_0^L f(s) \, ds$$ intial amount of energy.

Final amount.

$$E = Ac\rho \int_0^L D \, ds = Ac\rho DL$$

Thus

$$Ac\rho \int_0^L f(s) \, ds = Ac\rho DL$$

So $D = \frac{1}{L} \int_0^L f(s) \, ds$ where f is the initial distribution of heat in the rod.

Answer

$$u(x) = \frac{1}{L} \int_0^L f(s) \, ds$$

Finishes Section 1.4

Section 1.5: Heat equation in 3D, Read on your own for next time and review vector calculus...

Chapter 2: Separation of variables...

Heat equation: $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$

Define linear operator. $L = \dfrac{\partial}{\partial t} - k\dfrac{\partial^2}{\partial x^2}$

Thus $Lu = \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2}$

and the heat equation is written $Lu = 0$.

Linear means $L(u+v) = Lu + Lv$ (same as linear algebra)

$$L(u+v) = \frac{\partial(u+v)}{\partial t} - k\frac{\partial^2(u+v)}{\partial x^2} = \frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} - k\frac{\partial^2 u}{\partial x^2} - k\frac{\partial^2 v}{\partial x^2}$$
$$= Lu + Lv.$$

Note that $\frac{\partial u}{\partial t}$ may not exist when $\dfrac{\partial(u+v)}{\partial t}$ exists

For example consider any non-differentiable function u and take $v = -u$. Then

$$\frac{\partial(u+v)}{\partial t} = \frac{\partial(u-u)}{\partial t} = \frac{\partial 0}{\partial t} = 0$$

but $\frac{\partial u}{\partial t}$ doesn't exist. Warning to be careful.

Heat equation:

$$c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u(x,t)}{\partial x}\right) + Q(x,t)$$

heat capacity   density   conductivity.   rate of energy production inside the rod.

Let

$$Lu = c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u(x,t)}{\partial x}\right)$$

Then heat equation becomes

$$Lu = Q$$

note Q is not in definition of L because I want L to be linear.

Comparison with linear algebra.

if $Q=0$ then it's like $Ax=0$

if $Q\neq 0$ then it's like $Ax=b$