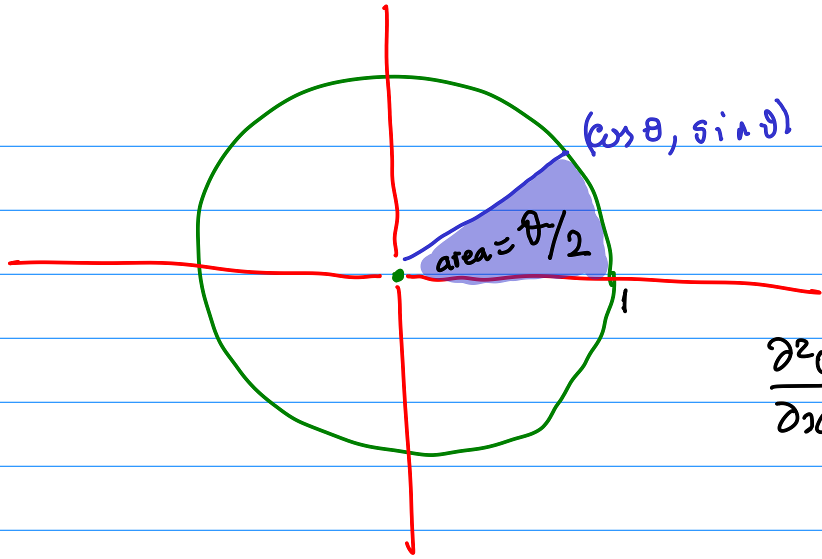
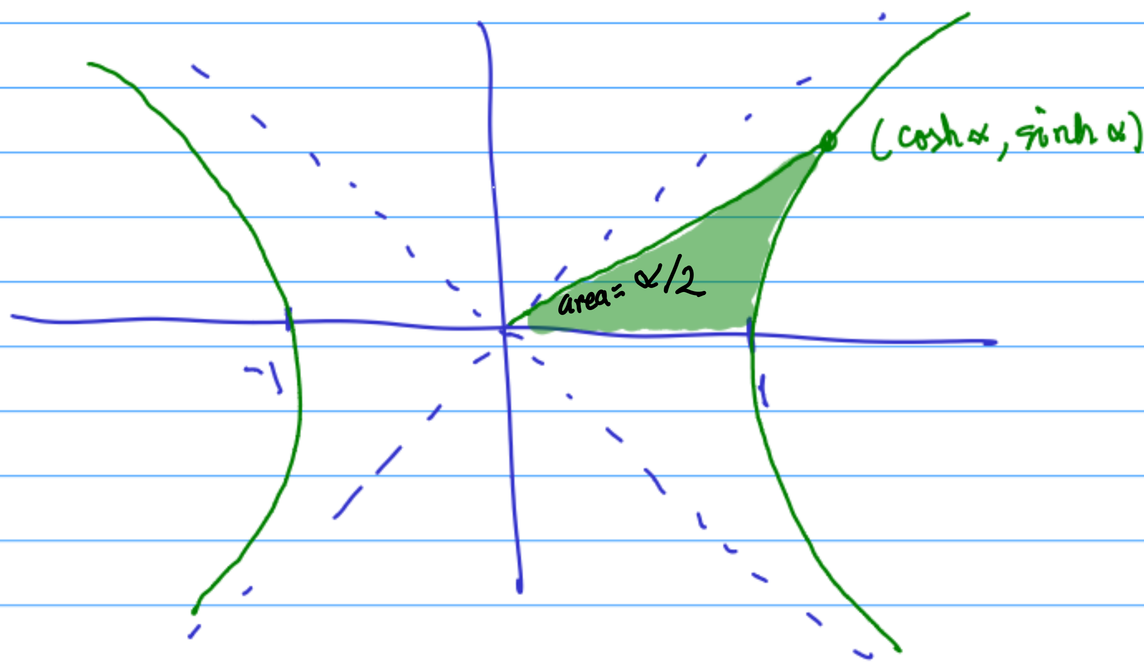


Polar coordinates

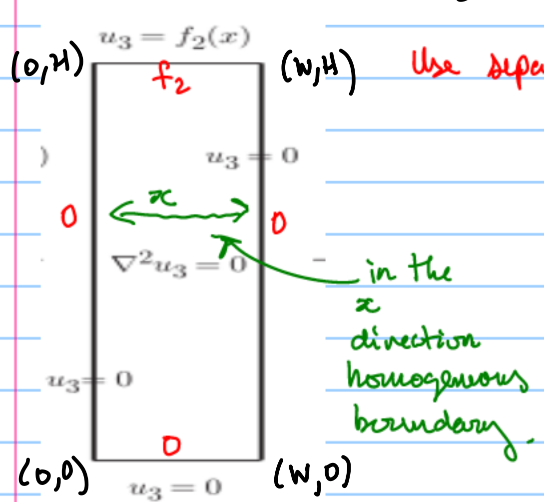
$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$



$u(x, H) = f_2(x)$



Use separation of variables..

$u(x,y) = \phi(x)\, w(y)$

identifies which direction has the homogeneous boundary.

$$\varphi(x) = C_2 \sin\left(\frac{n\pi}{W} x\right) \qquad \boxed{w(y) = C_2 \sinh \frac{n\pi}{W} y}$$

$$u(x,y) = \varphi(x) w(y)$$

Now use superposition to satisfy the remaining non-homogeneous boundary condition... $u(x, H) = f_2(x)$

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{W} x\right) \sinh\left(\frac{n\pi}{W} y\right)$$

Thus

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{W} x\right) \sinh\left(\frac{n\pi}{W} H\right) = f_2(x)$$

Use the orthogonality of the $\varphi(x)$ solutions to solve for A_n.

$$\sum_{n=1}^{\infty} \int_0^W A_n \underbrace{\left(\sin \frac{m\pi}{W} x\right) \sin\left(\frac{n\pi}{W} x\right)} \sinh\left(\frac{n\pi}{W} H\right) dx = \int_0^W \left(\sin \frac{m\pi}{W} x\right) f_2(x)\, dx$$

$$\int_0^W \left(\sin \frac{m\pi}{W} x\right) \sin\left(\frac{n\pi}{W} x\right) dx = \begin{cases} \frac{W}{2} & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

$$\frac{W}{2} A_m \sinh\left(\frac{m\pi}{W} H\right) = \int_0^W \left(\sin \frac{m\pi}{W} x\right) f_2(x)\, dx$$

$$A_m = \frac{2}{W \sinh(m\pi H/W)} \int_0^W \left(\sin \frac{m\pi}{W} x\right) f_2(x)\, dx$$

In summary

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{W}x\right)\sinh\left(\frac{n\pi}{W}y\right)$$

where

$$A_n = \frac{2}{W\sinh(n\pi H/W)}\int_0^W \left(\sin\frac{n\pi}{W}x\right) f_2(x)\,dx$$

is the solution to the Laplace equation

PDE $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

B.C.

$$u(x,0) = 0$$
$$u(0,y) = 0$$
$$u(W,y) = 0$$
$$u(x,H) = f_2(x)$$

Could call the solution to this problem u_3

B.C.

$$u(x,0) = f_1(x)$$
$$u(0,y) = 0$$
$$u(W,y) = 0$$
$$u(x,H) = 0$$

Call solution to these boundary conditions u_1

B.C.

$$u(x,0) = 0$$
$$u(0,y) = 0$$
$$u(W,y) = g_2(y)$$
$$u(x,H) = 0$$

Call this solution u_2

B.C.

$$u(x,0)=0$$
$$u(0,y)=g_1(y)$$
$$u(W,y)=0$$
$$u(x,H)=0$$

call this solution u_4

Finally the solution to

PDE $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

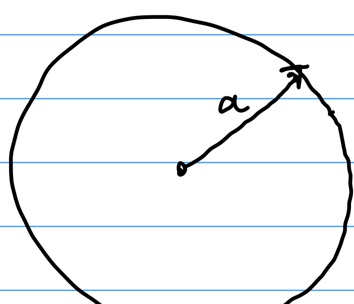
B.C.

$$u(x,0)=f_1(x) \leftarrow u_1$$
$$u(0,y)=g_1(y) \leftarrow u_4$$
$$u(W,y)=g_2(y) \leftarrow u_3$$
$$u(x,H)=f_2(y) \leftarrow u_2$$

each part of the boundary is handled by a different solution.

is obtained from $u(x,y)=u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)$

One more example. Laplace equation on a disk.



$u(a,\theta)=f(\theta)$ for $\theta\in[-\pi,\pi]$

radius, angle

polar coordinates

still linear so maybe superposition still works

PDE $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \qquad r\in[0,a],\ \theta\in[-\pi,\pi]$$

B.C.

$u(a,\theta)=f(\theta)$ $\quad\theta\in[-\pi,\pi]$ inhomogeneous

$\boxed{u(r,-\pi)=u(r,\pi)}$ $\quad r\in[0,a]$ homogeneous

$u_\theta(r,-\pi)=u_\theta(r,\pi)$

Let $u(r,\theta) = \varphi(\theta)\, G(r)$ Substitute it in

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\big(\varphi(\theta)\, G(r)\big)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\big(\varphi(\theta)\, G(r)\big) = 0$$

$$\varphi\, \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + \frac{G}{r^2}\frac{d^2\varphi}{d\theta^2} = 0$$

$$\frac{G}{r^2}\frac{d^2\varphi}{d\theta^2} = -\varphi\, \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)$$

$$\frac{1}{\varphi}\frac{d^2\varphi}{d\theta^2} = -\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\lambda$$

depends only on θ. depends only on r

Obtain two ODEs

$$\boxed{\begin{aligned} &\varphi'' = -\lambda\varphi \\ &\varphi(-\pi) = \varphi(\pi) \\ &\varphi'(-\pi) = \varphi'(\pi) \end{aligned}} \quad \text{and} \quad r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda G$$

Solve this first. General solution...

$\lambda > 0$ $\quad \varphi(\theta) = c_1 \cos\sqrt{\lambda}\,\theta + c_2 \sin\sqrt{\lambda}\,\theta$

$\lambda = 0$ $\quad \varphi(\theta) = c_1\theta + c_2$

$\lambda < 0$ $\quad \varphi(\theta) = c_1 \cosh\sqrt{\lambda}\,\theta + c_2 \sinh\sqrt{\lambda}\,\theta$

In which cases can I satisfy the boundary conditions with a non-zero solution?

~~Case~~ $\lambda > 0$

$$\varphi(\theta) = C_1 \cos\sqrt{\lambda}\,\theta + C_2 \sin\sqrt{\lambda}\,\theta$$

$$\varphi(-\pi) = \varphi(\pi)$$

$$\cancel{c_1 \cos\sqrt{\lambda}\pi} - c_2 \sin\sqrt{\lambda}\pi = \cancel{c_1 \cos\sqrt{\lambda}\pi} + c_2 \sin\sqrt{\lambda}\cdot\pi$$

Thus $\quad 2 c_2 \sin\sqrt{\lambda}\pi = 0$

Therefore $\sqrt{\lambda}\pi = n\pi \qquad n = 1, 2, \ldots$

$$\sqrt{\lambda} = n$$

$$\boxed{\lambda = n^2}$$