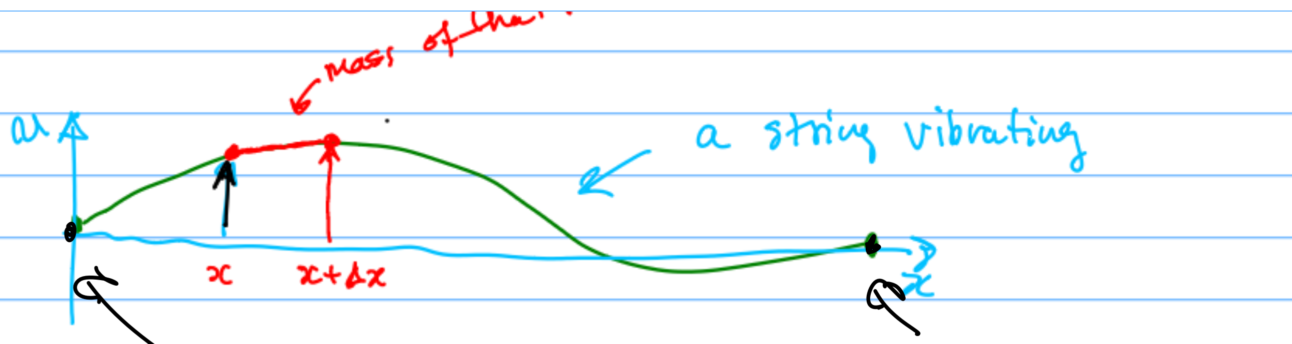
· Midterm on Friday March 15.
Covers up to section 3.6 (but not Chapter 4).

· Homework 4 due Wednesday.
I will turn off the upload link on Thursday and post my solutions.



$F = ma$

mass: $m = \rho_0 \Delta x$

acceleration: $a = \frac{\partial^2 u}{\partial t^2}$

Force from tension: $F = T(x+\Delta)\sin\theta(x+\Delta) - T(x)\sin\theta(x)$

Body force (gravity) (directly acting on the string): $F = \rho_0 \Delta x\, Q$

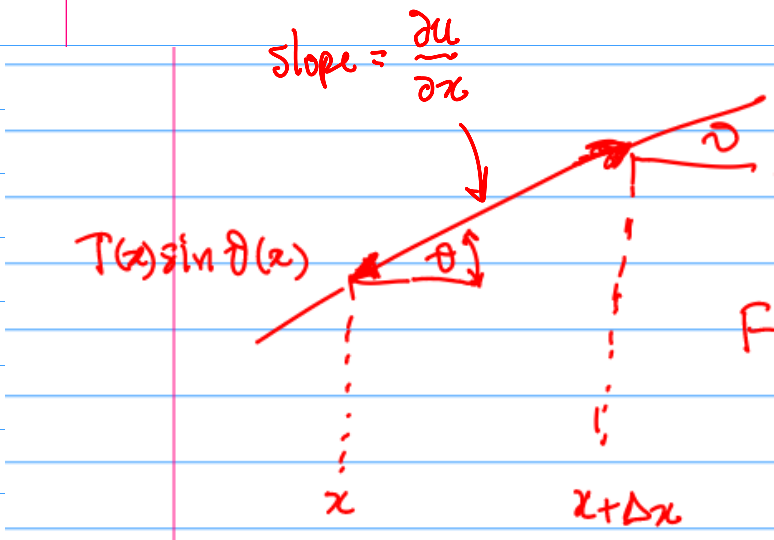
$$\rho_0 \Delta x \frac{\partial^2 u}{\partial t^2} = T(x+\Delta)\sin\theta(x+\Delta) - T(x)\sin\theta(x) + \rho_0 \Delta x\, Q$$

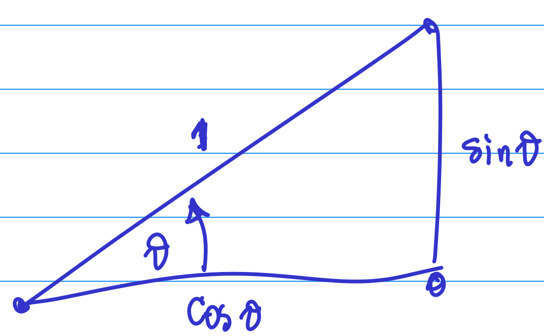
Take $\Delta x \to 0$ to obtain a PDE.

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{T(x+\Delta)\sin\theta(x+\Delta) - T(x)\sin\theta(x)}{\Delta x} + \rho_0 Q$$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} T(x)\sin\theta(x) + \rho_0 Q$$

want to express θ in terms of u





$$\frac{\partial u}{\partial x} = \text{slope} = \frac{\Delta Y}{\Delta X} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\theta = \arctan \frac{\partial u}{\partial x}$$

approximation of small oscillations...

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$\cos\theta \approx 1$ if θ is small..

$$\frac{\partial u}{\partial x} = \text{slope} \approx \frac{\sin\theta}{1}$$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} T(x) \sin\theta(x) \qquad + \rho_0 Q$$

Thus

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial u}{\partial x}\right) + \rho_0 Q$$

Wave equation

First order approximation of the physical problem under the assumption of small amplitude vibrations..

Assume tension is constant in the string. (again a first order approx)

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \rho_0 Q$$

or

$$\boxed{\frac{\partial^2 u}{\partial t^2}} = \boxed{c^2 \frac{\partial^2 u}{\partial x^2}} + Q \qquad \text{here } c^2 = \frac{T_0}{\rho_0}$$

Units: both terms need same unit of measurement

$$[u] = [L]$$

$$\left[\frac{\partial u}{\partial t}\right] = \frac{[L]}{[T]} \qquad \left[\frac{\partial u}{\partial x}\right] = \frac{[L]}{[L]} = 1$$

$$\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{[L]}{[T]^2} \qquad \left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{1}{[L]}$$

$$\frac{[L]}{[T]^2} = [c]^2 \frac{1}{[L]} \qquad [c]^2 = \frac{[L]^2}{[T]^2} \qquad [c] = \frac{[L]}{[T]}$$

a velocity.

Check that $\frac{[T_0]}{[\rho_0]} = [c]^2$

at home...

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q$$

Example suppose $Q = 0$ and then

PDE $\quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $t \geq 0$, $x \in [0, L]$

Boundary conditions $\quad u(0,t) = 0 \quad u(L,t) = 0$ for $t \geq 0$

homogeneous...

initial conditions $\quad u(x,0) = f(x)$ for $x \in [0,L]$

$\frac{\partial u}{\partial t}(x,0) = g(x)$ for $x \in [0,L]$

initial. velocity

Idea: separation of variables...

$u(x,t) = \varphi(x)\, h(t)$ plug it in...

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\varphi(x) h''(t) = c^2 \varphi''(x) h(t)$$

Thus

$$\frac{h''(t)}{c^2 h(t)} = \boxed{\frac{\varphi''(x)}{\varphi(x)} = -\lambda}$$

familiar with this part

Yields two ODEs

$$h''(t) = -c^2 \lambda h(t)$$

two initial conditions to satisfy by the superposition..

$$\boxed{\begin{gathered}\varphi''(x) = -\lambda \varphi(x) \\ \varphi(0) = 0 \quad \varphi(L) = 0\end{gathered}}$$

first

$$\varphi(x) = a \cdot \cos\sqrt{\lambda}\, x + b \cdot \sin\sqrt{\lambda}\, x$$

$$\varphi(0) = a = 0$$

$$\varphi(L) = b \sin\sqrt{\lambda}\, L = 0$$

So either $b = 0$ or $\underline{\sin\sqrt{\lambda}\, L} = 0$

means $\varphi = 0$ so no an eigenfunction

Thus $\sin\sqrt{\lambda}\, L = 0$ or $\sqrt{\lambda}\, L = n\pi$ or $\sqrt{\lambda} = \frac{n\pi}{L}$

Thus

$$\varphi_n(x) = b_n \sin\frac{n\pi x}{L}$$

Fourier sine series..,

Now solve the ODE for h.

$$h''(t) = -c^2 \lambda h(t)$$

General solution

$$h(t) = A\cos c\sqrt{\lambda}\, t + B\sin c\sqrt{\lambda}\, t$$
$$= A\cos\frac{cn\pi t}{L} + B\sin\frac{cn\pi t}{L}$$

Superposition

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n\cos\frac{cn\pi t}{L} + B_n\sin\frac{cn\pi t}{L}\right)$$

Solve for A_n and B_n
using the initial conditions ...

Initial condition

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n\cos\overbrace{\frac{cn\pi 0}{L}}^{1} + \cancel{B_n\sin\frac{cn\pi 0}{L}}\right)$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$\frac{\partial u}{\partial x}(x,0) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(-A_n\frac{cn\pi}{L}\cdot \quad - \quad \underline{\quad}\right)$$

whatever goes here

piecewise smooth g and smooth f means can diff. term by term.