PDE $\quad \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ $\quad$ for $t \geq 0$, $x \in [0, L]$

Boundary conditions $\quad \underbrace{u(0,t) = 0 \quad u(L,t) = 0}_{\text{homogeneous...}}$ $\quad$ for $t \geq 0$

initial conditions $\quad u(x,0) = f(x)$ $\quad$ for $x \in [0, L]$

$\dfrac{\partial u}{\partial t}(x,0) = g(x)$ $\quad$ for $x \in [0, L]$

Initial condition

using the initial conditions...

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{cn\pi 0}{L} + B_n \cancel{\sin\frac{cn\pi 0}{L}}\right)$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(-A_n\frac{cn\pi}{L} \cdot \quad - \quad \underline{\quad}\right)$$

whatever goes here

piecewise smooth g and smooth f means can diff. term by term.

Superposition

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}\right)$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{k=1}^{\infty} \sin\frac{n\pi x}{L}\left(-A_n\frac{cn\pi}{L}\sin\frac{cn\pi t}{L} + B_n\frac{cn\pi}{L}\cos\frac{cn\pi t}{L}\right)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{k=1}^{\infty} \sin\frac{n\pi x}{L}\left(-A_n \cancel{\frac{cn\pi}{L}\sin\frac{cn\pi 0}{L}} + B_n\frac{cn\pi}{L}\cos\frac{cn\pi 0}{L}\right)$$

$$g(x) = \sum_{k=1}^{\infty} B_n\frac{cn\pi}{L}\sin\frac{n\pi x}{L}$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$\int_0^L \sin\frac{m\pi x}{L} f(x)\,dx = \int_0^L \sum_{n=1}^{\infty} A_n \underbrace{\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}}\,dx$$

if $m \neq n$ they integrate to zero...

if $m = n$ $\int_0^L \sin^2 x\,dx = \frac{L}{2}$

$$\int_0^L \sin\frac{m\pi x}{L} f(x)\,dx = \frac{L}{2}A_m$$

$$A_m = \frac{2}{L}\int_0^L \sin\frac{m\pi x}{L} f(x)\,dx$$

$\sin a \cos b = \frac{1}{2}(\sin(a-b) + \sin(a+b))$

Angle addition

$$\sin(a+b) = \sin a\cos b + \cancel{\cos a \sin b}$$
$$\sin(a-b) = \sin a\cos b - \cancel{\cos a \sin b}$$

add

$$\frac{d}{da}\sin(a+b) = \frac{d}{da}(\sin a\cos b + \cos a \sin b)$$

$$\cos(a+b) = \cos a\cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a\cos b + \sin a \sin b$$

subtract

$$\cos(a+b) - \cos(a-b) = -2\sin a \sin b$$

$$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$

Superposition

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n\cos\frac{cn\pi t}{L} + B_n\sin\frac{cn\pi t}{L}\right)$$

$$u(x,t) = \sum_{k=1}^{\infty}\left(A_n \sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L}\right)$$

$$\sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L} = \frac{1}{2}\cos\left(\frac{n\pi}{L}(x-ct)\right) - \frac{1}{2}\cos\left(\frac{n\pi}{L}(x+ct)\right)$$

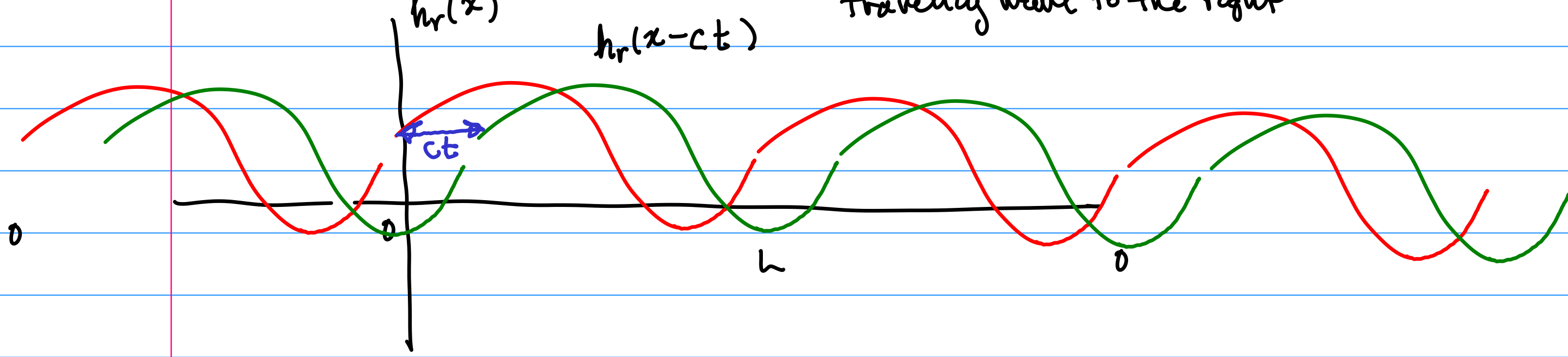
$$\sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L} = \frac{1}{2}\sin\left(\frac{n\pi}{L}(x-ct)\right) + \frac{1}{2}\sin\left(\frac{n\pi}{L}(x+ct)\right)$$

$$\sin a \cos b = \frac{1}{2}\left(\sin(a-b) + \sin(a+b)\right)$$

$$u(x,t) = \sum \frac{1}{2}\left(A_n \sin\left(\frac{n\pi}{L}(x-ct)\right) + B_n \cos\left(\frac{n\pi}{L}(x-ct)\right)\right)$$
$$+ \sum \frac{1}{2}\left(A_n \sin\left(\frac{n\pi}{L}(x+ct)\right) - B_n \cos\left(\frac{n\pi}{L}(x+ct)\right)\right)$$

$$u(x,t) = h_r(x-ct) + h_l(x+ct)$$ sum of two traveling waves..

traveling wave to the right



r

$$c^2 = \frac{T_0}{\rho_0}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q}$$

$$Q = -\frac{\beta}{\rho_0}\frac{\partial u}{\partial t}$$

4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

***(b)** Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$).

$$\underbrace{\boxed{\rho_0 \frac{\partial^2 u}{\partial t^2} = \cancel{T_0 \frac{\partial^2 u}{\partial x^2}}}}_{\text{energy conserving part}} \underbrace{- \beta \frac{\partial u}{\partial t}}_{\text{damping}}.$$

can't really cross of $\frac{\partial u}{\partial x^2}$ cause not even a PDE after that,

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = -\beta \frac{\partial u}{\partial t} \qquad v = \frac{\partial u}{\partial t}$$

$$\rho_0 \frac{\partial v}{\partial t} = -\beta v \qquad \frac{\partial v}{\partial t} = -\frac{\beta}{\rho_0} v$$

$$v(t) = c\, e^{-\frac{\beta}{\rho_0} t}$$

decay because of the $-\beta$.

Correct idea... identify an total energy and find how that energy changes in time...