PDE: $\boxed{\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2},}$

page 143

4.4.9. From $\boxed{(4.4.1)}$ derive conservation of energy for a vibrating string,

$$\boxed{\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L,} \tag{4.4.15}$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$.

$$E = \frac{1}{2}\int_0^L (u_t)^2\,dx + \frac{c^2}{2}\int_0^L (u_x)^2\,dx$$

have consist units of measurement.

Last time

$$\frac{d}{dt}\left(\int_0^L \rho_0 (u_t)^2\,dx + \int_0^L T_0 (u_x)^2\,dx\right)$$

$$E = \text{Kinetic energy} + \text{Potential energy.}$$

Similar boundary terms...

$$\boxed{u_x u_t \Big|_0^L} = u_x(L)\,u_t(L) - u_x(0)\,u_t(0)$$

Solution to HW:

$$E = \frac{1}{2}\int_0^L (u_t)^2\,dx + \frac{c^2}{2}\int_0^L (u_x)^2\,dx$$

$$\frac{d}{dt}E = \frac{d}{dt}\left(\frac{1}{2}\int_0^L (u_t)^2\,dx + \frac{c^2}{2}\int_0^L (u_x)^2\,dx\right)$$

$$= \frac{1}{2}\int_0^L \frac{\partial}{\partial t}(u_t)^2\,dx + \frac{c^2}{2}\int_0^L \frac{\partial}{\partial t}(u_x)^2\,dx$$

$$= \frac{1}{2}\int_0^L 2\, u_t \boxed{u_{tt}}\, dx + \frac{c^2}{2}\int_0^L 2\, u_x u_{xt}\, dx$$

Substitute

PDE: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2},$

$$= \frac{1}{2}\int_0^L 2\, u_t\, c^2 u_{xx}\, dx + \frac{c^2}{2}\int_0^L 2\, u_x u_{xt}\, dx$$

$$= c^2 \int_0^L (u_t u_{xx} + u_{xt} u_x)\, dx = c^2 \int_0^L \frac{\partial}{\partial x}(u_t u_x)\, dx$$

$$\frac{\partial}{\partial x}(u_t u_x) = u_t u_{xx} + u_{tx} u_x$$

by the fundamental theorem of Calculus

$$\frac{dE}{dt} = c^2 \int_0^L \frac{\partial}{\partial x}(u_t u_x)\, dx = c^2\, u_t u_x \Big|_0^L$$

Compare with this

$$\boxed{\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L,}$$

the same

Done with Chapter 4... HW will be assigned soon
please check the website.

CHAPTER 5

Sturm–Liouville Eigenvalue Problems

Self adjoint operators..

Linear partial differential equations.

We already know linear algebra...

Eigenvalue-eigenvector problem: Given $A \in \mathbb{R}^{n \times n}$ Solve

$$Ax = \lambda x \quad \text{for } x \text{ and } \lambda.$$

Review: ① This is a non-linear problem.

this term is quadratic

② Useful because converts matrix mult into scalar multiplication.

③ There is a theory: The spectral theorem.

Spectral Theorem. Let $A \in \mathbb{R}^{n \times n}$ such that $A^T = A$. (that is A is symmetric). Then there exist an orthonormal basis of eigenvectors, that is

$$Ax_i = \lambda_i x_i \quad \text{for } i = 1, \dots, n$$

and $x_i \cdot x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

It allows us to find the exponential of a matrix.

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots,$$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \dots,$$

• this is well defined since $A \in \mathbb{R}^{n \times n}$ so powers make sense.

Suppose $A = A^T$. The spectral theorem says there is a basis of orthonormal eigenvectors x_i for $i = 1, \dots, n$.

Let $Q = [x_1 | x_2 | \dots | x_n] \in \mathbb{R}^{n \times n}$ since $x_i \in \mathbb{R}^n$.

$$Q^T Q = \begin{bmatrix} x_1^T \\ \hline x_2^T \\ \hline \vdots \\ \hline x_n^T \end{bmatrix} [x_1 | x_2 | \dots | x_n] = \underbrace{[x_i^T x_j]}_{\text{general term of the matrix}} = [x_i \cdot x_j]$$

$$= \begin{bmatrix} x_1 \cdot x_1 & x_1 \cdot x_2 & \cdots & x_1 \cdot x_n \\ x_2 \cdot x_1 & x_2 \cdot x_2 & \cdots & x_2 \cdot x_n \\ \vdots & & \ddots & \vdots \\ x_n \cdot x_1 & x_n \cdot x_2 & \cdots & x_n \cdot x_n \end{bmatrix} = I$$

Thus $Q^{-1} = Q^T$ (the inverse exists and equal Q^T).

$$AQ = A[x_1 | x_2 | \dots | x_n] = [Ax_1 | Ax_2 | \dots | Ax_n]$$

$$= [\lambda_1 x_1 | \lambda_2 x_2 | \dots | \lambda_n x_n]$$

$$= [x_1 | x_2 | \dots | x_n] \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}}_{D}$$

$$= QD$$

Thus

$$AQ = QD \quad \text{or} \quad A = QDQ^{-1} = QDQ^T$$

What use is e^A? and how can I compute it?

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \dots$$

$$e^{QDQ^{-1}} = I + QDQ^{-1} + \frac{(QDQ^{-1})^2}{2!} + \frac{(QDQ^{-1})^3}{3!} + \dots$$

Note

$$(QDQ^{-1})^k = \underbrace{(QD\cancel{Q^{-1}})(\cancel{Q}D\cancel{Q^{-1}})\cdots(\cancel{Q}DQ^{-1})}_{k \text{ times}} = QD^kQ^{-1}$$

$$e^A = e^{QDQ^{-1}} = Qe^DQ^{-1} = Q\begin{bmatrix} e^{\lambda_1} & & 0 \\ & e^{\lambda_2} & \\ 0 & & \ddots \ e^{\lambda_n} \end{bmatrix}Q^T$$

since

$$(QDQ^{-1})^k = QD^kQ^{-1}$$

Then

$$e^{QDQ^{-1}} = I + QDQ^{-1} + \frac{(QDQ^{-1})^2}{2!} + \frac{(QDQ^{-1})^3}{3!} + \cdots$$

$$= I + QDQ^{-1} + \frac{QD^2Q^{-1}}{2!} + \frac{QD^3Q^{-1}}{3!} + \cdots$$

$$= Q\left(I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots\right)Q^{-1} = Qe^{D}Q^{-1}$$

We consider same Idea for PDEs.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} = Lu \qquad \text{where } L = k\frac{\partial^2}{\partial x^2}$$

Spectral theorem

claim is $L^{\dagger} = L$ (or L^*) so the eigenfunctions of L form an orthonormal basis

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Where did A^T come from?

$$y \cdot Ax = A^T y \cdot x$$

moving A over a dot product is where the transpose comes from.

Since dot products are replaced by integrals for PDEs then $L^{\dagger}$ has something to do with moving a differential operator from one term to the other in an integral. That is integration by parts.