12.6.2 Traffic Flow

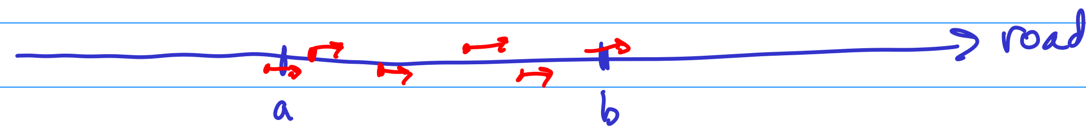
Traffic density and flow. As an approximation it is possible to model a congested one-directional highway by a quasilinear partial differential equation. We introduce the **traffic density** $\rho(x,t)$, the number of cars per mile at time t located at position x. An easily observed and measured quantity is the **traffic flow** $q(x,t)$, the number of cars per hour passing a fixed place x (at time t).

$\rho(x,t)$ cars/mile at point x and time t

$q(x,t)$ cars/hour passing by point x at time t (from left to right)

How does ρ change?

total # of cars in a certain stretch of road



$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = q(a,t) - q(b,t)$$

Let

$u(x,t)$ be the velocity of the cars at point x at time t

miles/hour.

Note that $q(x,t) = u(x,t)\,\rho(x,t)$

$$\frac{\text{cars}}{\text{hour}} = \frac{\cancel{\text{miles}}}{\text{hour}} \cdot \frac{\text{cars}}{\cancel{\text{mile}}}$$

Idea: Cars drive faster where fewer are on the road and slower when surrounded by lots of cars...

In the mid-1950s, Lighthill and Whitham and, independently, Richards made a simplifying assumption, namely, that the car velocity depends only on the density, $u = u(\rho)$, with cars slowing down as the traffic density increases (i.e., $du/dp \le 0$).

Thus ... $u(x,t) = u(\rho)$

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = q(a,t) - q(b,t) = -\int_a^b \frac{\partial}{\partial x} q(x,t)\,dx$$

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = u(a,t)\rho(a,t) - u(b,t)\rho(b,t)$$

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = \underbrace{u(\rho(a,t))\rho(a,t)} - \underbrace{u(\rho(a,t))\,\rho(b,t)}$$

function of ρ only...

Let $\psi(\rho) = \underbrace{u(\rho)\rho}_{\text{only fn of }\rho}$. thus. $q = \psi(\rho)$

$$\frac{\partial q}{\partial x} = \psi'(\rho)\frac{\partial \rho}{\partial x} = c(\rho)\frac{\partial \rho}{\partial x}$$

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = -\int_a^b c(\rho)\frac{\partial \rho}{\partial x}\,dx$$

$$\int_a^b \frac{\partial}{\partial t}\rho(x,t)\,dx = -\int_a^b c(\rho)\frac{\partial \rho}{\partial x}\,dx$$

or

$$\int_a^b \left[\frac{\partial}{\partial t}\rho(x,t) + c(\rho)\frac{\partial}{\partial x}\rho(x,t)\right]dx = 0$$

This holds for any stretch of road from a to b and therefore the integrand must be zero. That is,

$$\frac{\partial}{\partial t}\rho(x,t) + c(\rho)\frac{\partial}{\partial x}\rho(x,t) = 0, \qquad \rho(x,0) = f(x)$$

Let's try to solve this PDE using characteristics

$$x = x(t)$$

$$\frac{d}{dt}\rho(x(t),t) = \frac{\partial \rho}{\partial t} + x'(t)\frac{\partial \rho}{\partial x}.$$

Therefore we obtain the ODEs.

$$x'(t) = c(\rho) \qquad \text{and} \qquad \frac{d}{dt}\rho(x(t),t) = 0$$

$$\rho(x(t),t) = \rho(x(0),0) = f(x(0))$$

Let $x_0 = x(0)$

$$x'(t) = c\big(f(x(0))\big)$$

$$x(t) = t\,c(f(x_0)) + c$$

$$x(0) = 0 + c \qquad c = x_0$$

$$x(t) = \underbrace{t\,c(f(x_0)) + x_0}$$

← lines with a slope related to $c(f(x_0))$

Solution along the characteristic

$$\rho\big(t\,c(f(x_0)) + x_0,\ t\big) = f(x_0)$$

composition of functions ...

set $x = t\,c(f(x_0)) + x_0$ solve for x_0 :

Can it be done? How? and any intuition