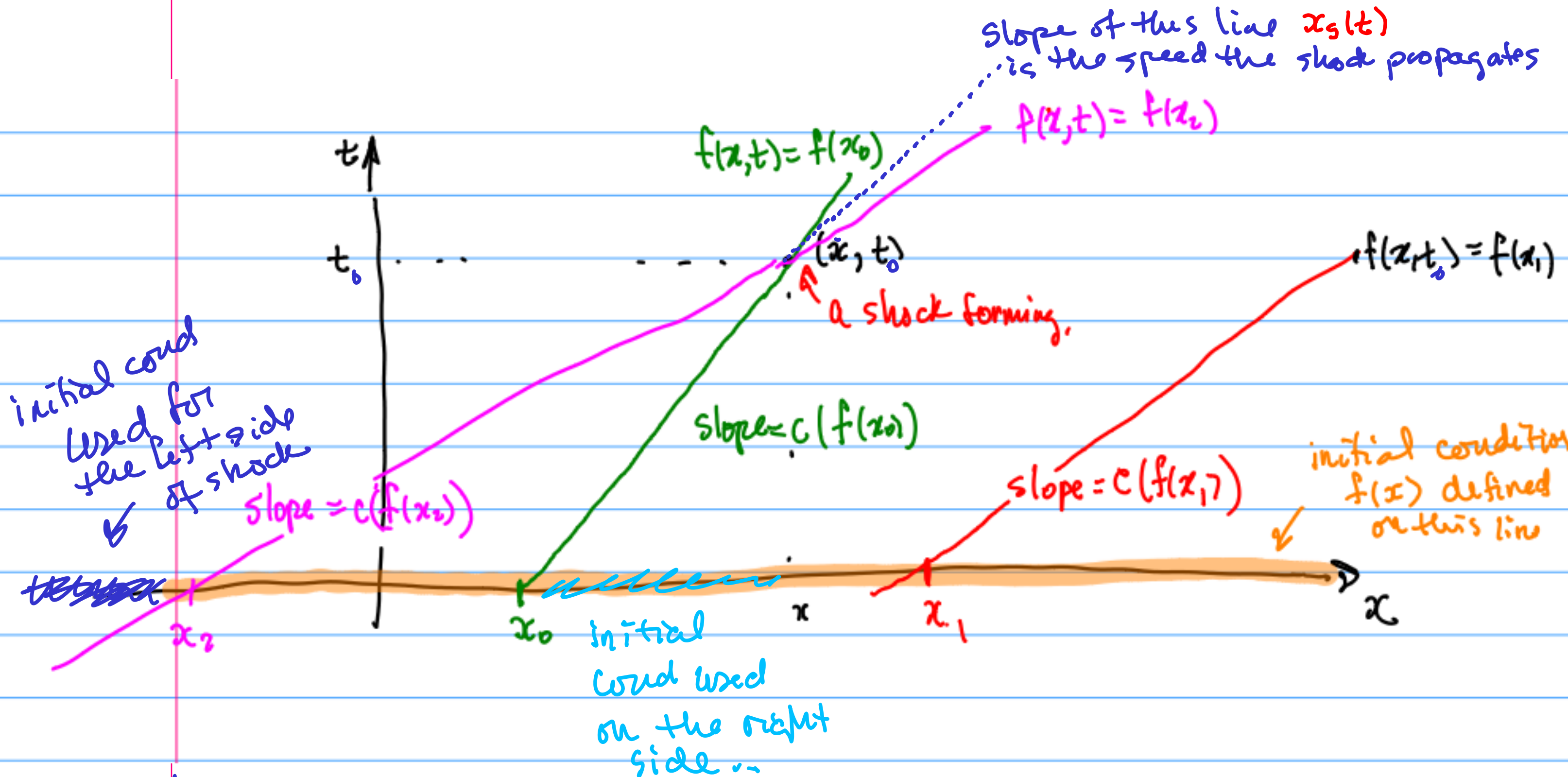

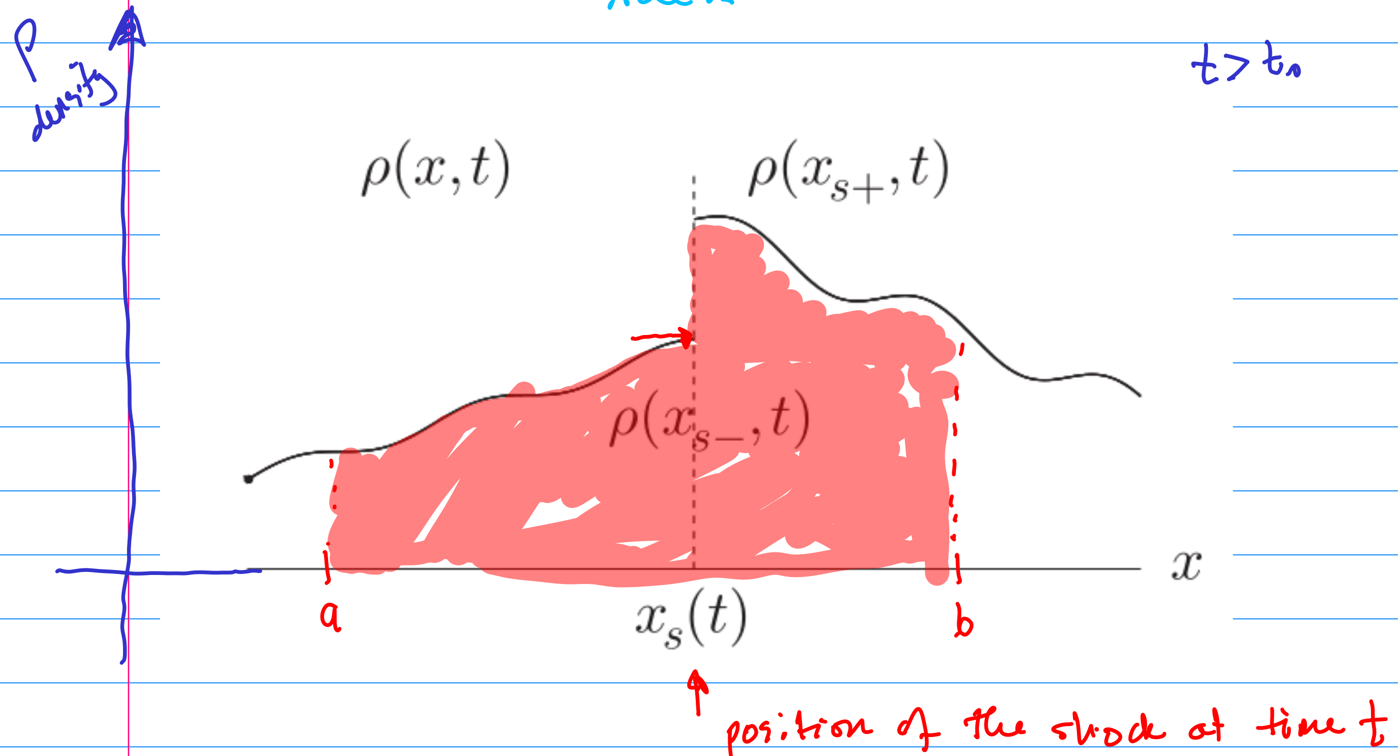

$$\frac{d}{dt}\boxed{\int_a^b \rho(x,t)\,dx} = q(a,t) - q(b,t)$$

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = \frac{d}{dt}\int_a^{x_s(t)} \rho(x,t)\,dx + \frac{d}{dt}\int_{x_s(t)}^{b} \rho(x,t)\,dx$$

$$\frac{d}{dt}\int_a^{x_s(t)} \rho(x,t)\,dx = \boxed{\frac{d}{dz}\int_a^{z} \rho(x,t)\,dx}\Bigg|_{z=x_s(t)} \frac{dx_s(t)}{dt} + \int_a^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx$$

$$= \rho(x_s(t)^-, t)\,\frac{dx_s(t)}{dt} + \int_a^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx$$

the density to the left of the shock.

$$\frac{d}{dt}\int_{x_s(t)}^{b} \rho(x,t)\,dx = -\frac{d}{dt}\int_b^{x_s(t)} \rho(x,t)\,dx.$$

$$= -\frac{d}{dz}\int_b^{z} \rho(x,t)\,dx\Bigg|_{z=x_s(t)} \frac{dx_s(t)}{dt} - \int_b^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx$$

$$= -\rho(x_s(t)^+, t)\,\frac{dx_s(t)}{dt} - \int_b^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx$$

density on right of shock.

$$= -\rho(x_s(t)^+, t)\,\frac{dx_s(t)}{dt} + \int_{x_s(t)}^{b} \frac{\partial}{\partial t}\rho(x,t)\,dx$$

Therefore

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = \underbrace{\rho(x_s(t)^-,t)}_{q}\frac{dx_s(t)}{dt} + \int_a^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx - \underbrace{\rho(x_s(t)^+,t)}_{q}\frac{dx_s(t)}{dt} + \int_{x_s(t)}^{b} \frac{\partial}{\partial t}\rho(x,t)\,dx$$

$$= q(a,t) - q(b,t)$$

Goal solve for $\frac{dx_s(t)}{dt}$ to see how the shock moves

$$\rho(x_s(t)^-,t)\frac{dx_s(t)}{dt} - \underbrace{\rho(x_s(t)^+,t)}_{q}\frac{dx_s(t)}{dt} = q(a,t) - q(b,t) - \int_a^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx - \int_{x_s(t)}^{b} \frac{\partial}{\partial t}\rho(x,t)\,dx$$

$$\frac{dx_s(t)}{dt} = \frac{q(a,t) - q(b,t) - \boxed{\int_a^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx - \int_{x_s(t)}^{b} \frac{\partial}{\partial t}\rho(x,t)\,dx}}{\rho(x_s(t)^-,t) - \rho(x_s(t)^+,t)}$$

Want to simplify the integrals using the PDE

$$\frac{\partial}{\partial t}\rho(x,t) + c(\rho)\frac{\partial}{\partial x}\rho(x,t) = 0,$$

$$\frac{\partial}{\partial t}\rho(x,t) = -c(\rho)\frac{\partial}{\partial x}\rho(x,t)$$

Thus

$$-\int_a^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx = \int_a^{x_s(t)} c(\rho)\frac{\partial}{\partial x}\rho(x,t)\,dx$$

assume velocity depends on ρ.

$$u(x,t) = u(\rho(x,t))$$

$$q(x,t) = u(x,t)\rho(x,t)$$

Let $\boxed{\psi(\rho) = \underbrace{u(\rho)\,\rho}_{\text{only fn of } \rho}}$ thus $q = \psi(\rho)$

$$\frac{\partial q}{\partial x} = \psi'(\rho)\frac{\partial \rho}{\partial x} = c(\rho)\frac{\partial \rho}{\partial x}$$

$$\frac{\partial q}{\partial x} = c(\rho)\frac{\partial \rho}{\partial x}$$ from the derivation

Since the PDE holds on each side of the shock

$$-\int_a^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx = \int_a^{x_s(t)} c(\rho)\frac{\partial}{\partial x}\rho(x,t)\,dx = \int_a^{x_s(t)} \frac{\partial q}{\partial x}\,dx = q(x_s(t)^-) - q(a)$$

$$-\int_{x_s(t)}^{b} \frac{\partial}{\partial t}\rho(x,t)\,dx = \int_{x_s(t)}^{b} c(\rho)\frac{\partial}{\partial x}\rho(x,t)\,dx = \int_{x_s(t)}^{b} \frac{\partial q}{\partial x}\,dx = q(b) - q(x_s(t)^+)$$

$$\frac{dx_s(t)}{dt} = \frac{\cancel{q(a,t)} - \cancel{q(b,t)} + q(x_s(t)^+) - \cancel{q(a)} + \cancel{q(b)} - q(x_s(t)^-)}{\rho(x_s(t)^-,t) - \rho(x_s(t)^+,t)}$$

$$\frac{dx_s(t)}{dt} = \frac{q(x_s(t)^+) - q(x_s(t)^-)}{\rho(x_s(t)^-,t) - \rho(x_s(t)^+,t)}$$ ← somehow there is a sign error in the numeration,

$$\boxed{\frac{dx_s}{dt} = \frac{q(x_{s-},t) - q(x_{s+},t)}{\rho(x_{s-},t) - \rho(x_{s+},t)} = \frac{[q]}{[\rho]},}$$